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# GEORGE ELLERY HALE
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Dr. George Ellery Hale, distinguished and much beloved astronomer, died in Pasadena, California, on February 21 after an illness extending over several years. A scientist who made many remarkable contributions to his chosen field of solar research, the founder and almost throughout his life an editor of the *Astrophysical Journal*, founder and first director of the Yerkes and the Mount Wilson observatories, organizer of the International Union for Co-operation in Solar Research and of its successor, the International Astronomical Union, for many years foreign secretary of the National Academy of Sciences, and the founder of the National Research Council, Dr. Hale has left an impress upon the life of science which will endure for generations to come. In later years, when illness prevented him from carrying on actively the investigations he had most at heart, he conceived the project of the 200-inch telescope and secured the necessary financial support, leaving this great instrument as his final gift to the science he had loved so well.

The name of Hale is associated with so many important organizations in the creation and development of which he took the leading part that it is not always easy to realize that the instinct for research was the dominant feature of his life. He was a born investigator and was never so thoroughly content as when he could work

uninterruptedly at the telescope and spectrograph or could analyze his results in the quiet of his office or laboratory. For this reason there can be little doubt that the happiest years of his life were those at Kenwood and the earlier years at the Yerkes and Mount Wilson observatories. They were periods of construction and development and gave full scope to his creative and experimental ability before the heavy responsibilities of the larger organizations had begun to be felt to any marked degree. But Hale also realized, to an extent much greater than most investigators, that the major problems of physical science are usually so extensive and complex that progress in their solution requires the co-operative effort of many individuals and many institutions. So, although by nature a strong individualist in his scientific work and in his outlook on life, he was constantly engaged in organizing plans for co-operative observations, in encouraging young investigators, and in securing the establishment of new institutions and wider and better facilities for research.

Of the conduct of his scientific work it would hardly be possible for any two, even of Hale's closest, associates to write in the same way, such were the characteristics of this gifted and many-sided man. His active and brilliant mind was constantly at work analyzing his problems, weighing new hypotheses, and designing new methods of attack, always with an enthusiasm resembling the creative joy of the artist. He had to a remarkable degree the gift of being able to distinguish the essential and salient features of an investigation; and it was rarely, indeed, that even a minor research which he attempted failed to bring productive results. One very striking characteristic was his interest in instruments and his remarkable skill in the designing of new apparatus. The invention of the spectrohe'liograph and the spectrohelioscope and the adaptation of the tower telescope and the vertical spectrograph for solar investigation are excellent examples of this quality of Hale's mind. In part it was probably due to his engineering training, but much more to his appreciation of the enormous importance of the value of improved instruments to astronomical progress, and to his creative ability and love of fine workmanship. He took immense pleasure in seeing a new instrument assume form; and, as he often said,

"had I not been an astronomer, I should have liked to have been an instrument-maker."

The early years of Hale's scientific life and development form an exceedingly interesting story. We may almost say that, as in the case of Michelson, the phenomena of light fascinated him from the beginning. From childhood the microscope and telescope, the colors of the spectrum, and the infinite variety of forms in which light manifests itself in everyday life were an unfailing source of joy and satisfaction to him. Probably no boy ever used a diffraction grating at so early an age, and the story of Hale's first meeting with Rowland and the latter's remark that "any grating of mine should be good enough for an infant" was always quoted with much amusement by Hale in after-life. His father gave him constant encouragement in these interests, and gradually the equipment was built up which came to form the Kenwood Observatory.

Fortunately, we have in Hale's own words a most interesting statement regarding this formative period of his life and some of the influences which led him to take up astrophysical research as a career in after years. The statement is contained in some biographical notes, written by him in 1933, and represents his mature judgment on the factors which affected most greatly the direction of his intellectual interests. The notes also throw an amusing sidelight on one or two astronomical worthies of this period.

I cannot fix the date of my first interest in astronomy, but it must have been when I was thirteen or fourteen years old. I built a telescope; but as I used a large single lens, the images were not good. About this time I became acquainted with S. W. Burnham, then a stenographer in the Chicago law courts by day and an ardent observer of double stars by night. Through him I learned of a second-hand Clark refractor, of 4 inches aperture. This was purchased by my father, and I mounted it on the roof of the house. The astonishing views it afforded of Saturn, the moon, Jupiter, and other objects excited an intense desire to carry on actual research. So I attached a plateholder to the telescope and photographed a partial eclipse of the sun. I also began to observe sunspots, and made drawings of them. Thus I became an amateur astronomer.

At this period the 18½-inch Dearborn refractor stood at the summit of a lofty tower forming part of the castellated building of the old University of Chicago (long since demolished, after the removal of the telescope to the Northwestern University at Evanston). Professor G. W. Hough, known for his observations of Jupiter and double stars, who was in charge of the instrument,

was kind enough to let me look through it frequently. I was also permitted to aid the feeble gas-engine in turning the dome, or rather drum, of the observatory. Naturally, my own ambitions were thus stimulated, but neither Hough nor Burnham had the slightest interest in astrophysical research, and I could not have devoted my life to such work as they were doing, valuable as it was to science. The reason lay in the fact that I was born an experimentalist, and I was bound to find the way of combining physics and chemistry with astronomy. Fortunately, it was not far to seek.

In Cassell's book is a description of a spectroscope, with an account of the construction of a simple instrument, using either a luster prism or a hollow one filled with carbon disulphide. I built both forms, and the odor of the disulphide abides with me after a lapse of fifty years. Then my father, always ready to encourage serious efforts, enabled me to buy a small spectrometer. This had a single prism, and I lost no time in fitting it with a small plane grating. For by this time I had learned of the work of Rutherfurd and Rowland and had acquired some slight conception of the possibilities of high dispersion. Nothing could exceed my enthusiasm in observing the solar spectrum and in measuring the principal lines. I bought Lockyer's *Studies in Spectrum Analysis* and began the observation of flame and spark spectra and their comparison with the spectrum of the sun. At last I had found my true course, and I have held to it ever since.

In passing, however, I must emphasize the advantages I gained from my earlier work in other fields. It gave me a broad interest in many branches of science and taught me something of their mutual dependence. It led me to read *The Origin of Species* and initiated a lasting desire to study evolution. It taught me to regard the sun as a typical star, a link in the long evolutional chain, and thus helped me to avoid becoming exclusively a specialist in solar research. And it was the source of all the work I have done in helping to develop such institutions as the National Academy of Sciences, the California Institute of Technology, the National Research Council, and the International Research Council (now the International Council of Scientific Unions). I still look with delight at rotifers and other microscopic objects, and realize that the marvels I saw through the little microscope of my boyhood days are as astonishing as any revealed in the heavens by the largest of telescopes.

The early education which Hale received in Chicago was at the Oakland Public School and later at Allen Academy, where he was allowed to care for the "philosophical instruments" and to assist in demonstrations before the student body. At the suggestion of Daniel H. Burnham, the well-known architect, who was a close friend of Hale's father, the Massachusetts Institute of Technology was chosen for the college career of the son. Hale entered the Institute in the autumn of 1886 and was graduated in 1890. His courses

were mainly in mathematics, physics, and chemistry, and for astronomy he was obliged to go elsewhere. Through special arrangement with E. C. Pickering he was able to serve as volunteer assistant at the Harvard College Observatory, where, among other experimental instruments, he tried his first form of spectroheliograph.

Hale's marriage to Miss Evelina Conklin occurred soon after he left the Massachusetts Institute, and on their western trip together they visited the Lick Observatory, where Hale was greatly impressed by the 36-inch telescope and the admirable spectroscopic investigations of planetary nebulae by Keeler. On his return to Chicago he passed three active years of research at the Kenwood Observatory, interrupted only by a trip in 1891 to Europe, where he met many astronomers and spectroscopists and studied their methods and equipment. The winter of 1893–94 he spent in study at the University of Berlin, where the lectures by Planck on thermodynamics and theories of radiation were of great value to him in his later work. He made frequent visits to Potsdam and became thoroughly familiar with the researches of Vogel, Scheiner, and others. It was during this European journey that he completed his plans for the establishment of the *Astrophysical Journal.*

After his stay in Berlin and an expedition to Mount Aetna, where he tried out his apparatus for photographing the corona without an eclipse, he returned to Chicago, where he continued his research at the Kenwood Observatory but was also actively engaged in plans for the larger institution which had taken form as the Yerkes Observatory.

The University of Chicago was at this time in its great period of development under President Harper. A remarkable judge of men, he at once appreciated the abilities of this gifted student and investigator of twenty-four, and Hale became a member of the staff of the University as associate professor of astrophysics in 1892. Together they formed plans for a great observatory; and, both being men of initiative and constructive imagination and enthusiasm, they succeeded in interesting Mr. Yerkes of Chicago, who made the gift for the project in the same year. Hale carried on the detailed plans with characteristic energy, outlining the investigations to be undertaken, designing instruments, and assembling a staff. Barnard, with

his wide experience and extraordinary skill in visual observations, came from Mount Hamilton; Burnham, also with Lick Observatory training, from Chicago; Wadsworth, from Michelson's laboratory; Frost, from Dartmouth College; and Ellerman, with his background of years of experience with Hale, from Kenwood. The Yerkes Observatory was completed and dedicated in 1897, with Hale as its first director and professor of astrophysics at the University of which it formed a part. He retained his relationship to the University of Chicago even after he came permanently to California, being nonresident professor from 1905 onward.

The winter of 1903–4, owing to illness in his family, Hale spent in Southern California. His attention had previously been attracted to Mount Wilson by Professor Hussey, who was investigating possible sites for a high-altitude solar observatory in behalf of a committee appointed by the Carnegie Institution of Washington, of which Hale was a member. In June, 1903, accompanied by Dr. W. W. Campbell, Hale made his first visit to Mount Wilson, where Hussey was still continuing his observations. Tests of the solar seeing were made by Hale at this time and during several of the succeeding winter months. They proved so satisfactory that application was made to the Carnegie Institution for a grant to finance an expedition from the Yerkes Observatory and the installation of the Snow horizontal telescope. A group consisting of Ellerman, Ritchey, and Adams came in the spring of 1904, followed somewhat later by F. G. Pease, and formed the nucleus of the staff of the future Mount Wilson Observatory. Throughout 1904, systematic records of both solar and stellar observing conditions were maintained; and, after much careful consideration, Hale recommended to the Carnegie Institution the establishment of an independent observatory. A grant for this purpose was made in December, 1904; the Snow telescope was purchased from the Yerkes Observatory; and the Mount Wilson Solar Observatory, later changed to the Mount Wilson Observatory, as its stellar activities increased, was organized.

This account of the origin of the Mount Wilson Observatory is given somewhat in detail to illustrate the caution and painstaking care which Hale combined with his enthusiasm in the development of his major projects. Probably no observatory was ever established

after so thorough an examination of observing conditions and of all the physical considerations entering into its construction, maintenance, and growth. Questions of transportation, water supply, electric power, building materials, and living conditions were all investigated by him personally; and no subject was too small to be given thought and consideration. It was a period in which he enjoyed life enormously. The isolation of the mountain top in these early years made a strong appeal to the pioneering spirit latent in him, and his joy in discoveries of the everchanging beauties of nature about him was second only to that of his discoveries in science.

In 1910, after six years of active and most productive work, the first shadows of physical illness began to fall across Hale's life. After this time he was never completely free from more or less severe attacks of a type of brain congestion which occasionally produced severe pain and almost complete mental exhaustion. He soon learned that such attacks were most apt to occur when his mind was occupied with his scientific work, and with profound courage and at great sacrifice he attempted to adjust himself to the situation. Without completely giving up his research, he devoted more time to writing, to executive work, and to the organization and development of projects he had especially at heart. After the World War broke out, he spent much time in Washington, where, at the request of President Wilson to the National Academy of Sciences, he was mainly responsible for the creation and organization of the National Research Council. He was foreign secretary of the National Academy from 1910 to 1921 and gave much thought to plans for making the Academy a more active force in national and international science. He also visited Europe several times to attend meetings of the International Research Council, afterward the International Council of Scientific Societies, and aided greatly in directing its policies. Many of these interests he maintained to the end of his life.

On his return to the Mount Wilson Observatory, Hale continued his scientific work as his strength permitted, but with increasing difficulty; and in 1923 he resigned the directorship. He was appointed honorary director by the Carnegie Institution; and, with full realization of his physical limitations and with remarkable courage,

he faced the problem of how to make the most effective use of such years as remained to him. He solved this problem in two ways. First, he constructed, largely at his own expense, the small, well-equipped solar laboratory which he presented to the Carnegie Institution, and in which he could carry on such research as his health allowed. Second, he undertook the writing of a number of articles planned to bring some of the results of astronomical discovery before the intelligent public. In this purpose he was remarkably successful, his clarity and simplicity of statement forming an admirable setting for the spirit of enthusiasm pervading all his writing.

Two striking, but widely dissimilar, accomplishments mark these later years of Hale's life. The first was the invention of the spectrohelioscope, which supplements in many important respects the spectroheliograph devised by him so many years ago at Kenwood. It is already yielding results of great value at stations all over the world and, through the application of moving-picture attachments, gives promise of proving the most efficient instrument yet designed for the study of rapidly changing phenomena on the sun's surface. The second was the project of the 200-inch telescope. Attracted by his article published in 1928, "The Possibilities of Large Telescopes," and strongly influenced by their personal knowledge of Hale and his contributions to science, the president and trustees of the General Education Board made a grant for the construction of a 200-inch telescope, the instrument being given to the California Institute of Technology on the basis of a plan of co-operation with the Carnegie Institution and the Mount Wilson Observatory. From 1929 until the closing days of his life, problems relating to this great instrument occupied most of Hale's thoughts. He was chairman of the group having the project in charge, and his active mind was fertile in suggestions for the design both of the telescope itself and of the equally important auxiliary instruments. As illness increased and his strength gradually failed, his mind turned constantly to the progress of the telescope and the observatory; and each clear day, as the winter rains during the past winter were delayed, he rejoiced in "one more day for work on Palomar."

The first article published by Hale appeared in 1889 in a small Chicago periodical, *The Beacon*, and was entitled "Stellar Photog-

raphy." This was written at the age of twenty. Succeeding papers during this period appeared in *The Beacon*, *Astronomische Nachrichten*, *American Journal of Science*, *Sidereal Messenger*, *Publications of the Astronomical Society of the Pacific*, and especially *Astronomy and Astrophysics*, the astrophysical portion of which he edited during the three years of its existence. He had for several years considered the desirability of establishing a journal devoted to spectroscopy and astrophysics and had sought carefully the opinions of scientists from all parts of the world. Assured of their strong support, Hale founded in 1895, in collaboration with James E. Keeler, the *Astrophysical Journal*, the University of Chicago assuming the responsibilities of publication and taking over, through purchase, *Astronomy and Astrophysics*. The new journal was successful from the beginning and was truly international in character, with a board of collaborators whose names were eminent in science throughout the United States and Europe. Many of the classical papers of astrophysics and spectroscopy have first appeared in the *Astrophysical Journal*, and the influence of its founder and first editor has been maintained throughout the period of its existence, now nearly half a century in length.

Nothing illustrates more clearly Hale's breadth of interests than a study of his bibliography. In a list of nearly four hundred and fifty titles there is found a range from spectroscopic observations of solar prominences to an analysis of the intellectual culture of France. Included in it are his two interesting nontechnical books, *Stellar Evolution* and *Ten Years' Work of a Mountain Observatory*, together with compilations, in book form, of his numerous magazine articles of an interpretive character. The longer technical monographs appeared in the publications of the Yerkes Observatory and the Carnegie Institution of Washington, but most of his best-known papers are found in the *Astrophysical Journal*. About 90 per cent of his published articles deal with the results of his scientific researches.

The earliest work which Hale carried on at the Kenwood Observatory consisted of visual observations of solar prominences. From the outset he realized the immense value of photography in affording a complete and permanent record of these objects, and

his ingenious mind immediately undertook to accomplish this purpose. The problem was twofold in character. First it was necessary to find a spectral line, other than the hydrogen *Hα* used for visual observations, through which the prominences could be photographed, since the photographic plates of this period had little sensitiveness to red light. Second, an instrument must be devised by means of which the small portion of the image of the prominence or the sun's disk which appeared in the light of the spectral line could be made to build up the complete image. Hale solved the first part of the problem through his investigations of the H and K lines of calcium, already known through the work of C. A. Young, to be always bright in the chromosphere and prominences, as well as over sunspots. These lines, which Hale was the first to identify with spark lines of calcium, were used for most of his work with the spectroheliograph until 1908, when photographic plates more sensitive to red light became available and the *Hα* line could also be used to advantage. It is probable, however, that he never felt inclined to revise his early statement about the H and K lines, "which to my mind form the most interesting group in the solar spectrum."

Two instrumental devices for obtaining photographic records of prominences and other solar phenomena were invented by Hale as early as 1889. In the second of these, which he finally adopted at the Kenwood Observatory, and to which he gave the name of "spectroheliograph," two movable slits were used, driven by a specially designed clepsydra, one across the image of the sun and the other immediately in front of the photographic plate. The instrument was extraordinarily successful from the outset, and for years Hale's publications described the new and remarkable results obtained with it. Although his three attempts to photograph the corona with the spectroheliograph in full sunlight—at Kenwood, Pike's Peak, and Mount Aetna—proved unsuccessful, they form an admirable illustration of his characteristic determination to carry the application of an instrument to its extreme limit, a quality which proved brilliantly successful on so many occasions. Nearly all our existing knowledge regarding the calcium and hydrogen flocculi, the forms and motions of prominences, details in the chromosphere, and the structure of the solar surface around sunspots is due to the spectroheliograph or to some adaptation of its principles.

During this period Hale also carried on some extremely valuable investigations of the spectra of prominences and sunspots, identifying the ultraviolet lines of the Balmer series of hydrogen and numerous lines of other elements in active prominences, measuring distortions of the *Hα* line and studying its behavior relative to H and K, and observing with great care the $D_3$ line of helium both as a bright line in the spectrum of the chromosphere or prominences and occasionally as an absorption line near active sunspots. The spectrum of sunspots already interested him greatly, and it is quite probable that he would have anticipated some of his later discoveries could his spectroscopic apparatus, necessarily attached to a small refracting telescope, have had sufficient resolution. Theories of sunspots and of the relationship of faculae and flocculi to each other and to spots attracted his attention, and he was greatly influenced by the work of Secchi and Tacchini in Italy and of Faye in France. An interesting summary of many of the results and conclusions of his work at this time was published in October, 1892, in *Astronomy and Astrophysics*.

As Hale moved onward to the wider field of work afforded by the Yerkes Observatory, he saw opportunities for extending his investigations with more powerful apparatus. A large spectroheliograph was one of the first requisites, and the Rumford spectroheliograph, when attached to the 40-inch refractor, has remained one of the most efficient instruments for certain types of solar research ever designed. With it dark hydrogen flocculi were discovered and the dark calcium markings which proved to be prominences seen in projection against the disk. The dispersion of the instrument was sufficient to make it possible to set the second slit upon different portions of the K line, and thus to map the calcium flocculi at different levels throughout the solar atmosphere. A grating spectrograph supplemented the spectroheliograph, and with it Hale made the difficult visual discovery of the presence of the green carbon band in the spectrum of the chromosphere. He became fully aware, at this time, of the possibility of observing the chromospheric and much of the "flash" spectrum without an eclipse, and in future years continued such work at Mount Wilson.

A strangely prophetic remark was made by Hale in an article published at the Yerkes Observatory in 1902: "The remarkable

peculiarities of the spectra of sunspots seem to deserve more attention than they have hitherto received from spectroscopists." Widened lines in the spectra of spots had been observed visually by Young in 1872 and were studied systematically by the English astronomers Maunder, Cortie, and Lockyer. Hale made some attempts to photograph the spectrum of sunspots while at the Kenwood Observatory, but his success was only partial because of the small size of the sun's image. With the larger image given by the 40-inch telescope at the Yerkes Observatory much better results were obtained, and not only were the widened lines measured but also many of the lines in a green band discovered by Maunder in 1880. Hale realized that for an adequate investigation of the spot spectrum greater linear scale in the spectrograph and a still larger solar image were needed. Accordingly, he designed for the purpose a large spectrograph used in conjunction with a coelostat and long-focus concave mirror, thus planning to bring into regular astronomical use an instrument which had been relegated almost wholly to eclipse expeditions. This instrument finally took form in the Snow telescope, with which in later years at Mount Wilson the first of the extensive sunspot investigations was begun.

Although Hale's primary interest in life lay in the study of the sun, he always emphasized the fact that the sun is an individual star, so close that it can be studied in detail, but only one object in the vast physical universe about us. For any adequate study of the major problems of astronomy, therefore, he felt it necessary to have both solar and stellar equipment on an extensive scale, supplemented by a physical laboratory to which recourse could be had for the interpretation of astrophysical phenomena. This principle underlay his plans for each of the observatories he established, and so we find great stellar telescopes at Williams Bay, Mount Wilson, and on the summit of Palomar. That he developed this point of view at an early age and at the same time fully appreciated the striking value of that somewhat neglected instrument, the reflecting telescope, is shown by the gift, through his father in 1895, of the 60-inch mirror to the Yerkes Observatory. This is now incorporated into the 60-inch reflector on Mount Wilson.

The 40-inch telescope at the Yerkes Observatory gave Hale his

first opportunity to study stellar spectra. The objects selected were the stars of Secchi's Fourth Type (Harvard type N), which heretofore had been observed only visually. The carbon bands and a very few prominent lines had been recognized, but otherwise the spectra were essentially unknown. Observational difficulties were great, since all the stars were faint and red and red-sensitive photographic plates were very much in their infancy. With the active assistance of Ellerman and J. A. Parkhurst, however, Hale obtained a series of photographs remarkably good for this period of instrumental development, and discussed the results in detail. The lines of numerous elements were identified in the spectra, bright lines were observed in several stars, a very significant resemblance to certain features of sunspot spectra was discovered, and the physical condition of the stars and their place in the sequence of stellar development were considered from numerous points of view. With but one or two exceptions the conclusions drawn in this important investigation remain without change to the present time, later studies with more powerful apparatus supplementing, rather than modifying, them in any essential way.

Hale's great interest in laboratory investigations and his constant desire to apply physical tests to astrophysical phenomena led him, while at the Yerkes Observatory, to undertake, in collaboration with N. A. Kent, an extensive research on the spectrum of the iron spark in liquids and compressed gases. At first he thought the discovery of lines consisting of both emission and absorption components might have application to the phenomena of novae, but later he gave up this view. As a physical investigation, however, it was important in its bearing on the separation of lines according to degree of excitation, the displacement of lines under pressure, and the study of selective absorption as a function of wave length.

The establishment of the Mount Wilson Observatory gave Hale, for the first time, the opportunity of building instruments on a large scale to fit the problems to be investigated rather than of finding the problems to be undertaken with an existing instrument. This is the point of view of the physicist and was a principle maintained by Hale throughout his life. The Snow telescope was designed for certain specific purposes, primarily to make possible the

use of larger spectrographs and the violet region of the spectrum; the 60-foot tower telescope, to improve seeing conditions on the sun's image and afford convenience of operation and temperature control for very long spectrographs; and the 150-foot tower telescope, to afford a very large solar image for the study of sunspots and details on the sun's surface. The same attitude was taken with reference to the stellar instruments, the 60-inch and 100-inch telescopes being designed for use at three foci, so that the magnification and the auxiliary instruments could be adjusted to the problem in mind. Hale was always greatly interested in the coudé form of the reflecting telescope, appreciating from experience the value of large-scale stellar spectra and the immense advantages of fixed instruments operated under laboratory conditions.

The earliest of Hale's publications from Mount Wilson are largely descriptive, dealing with observing conditions, a subject in which he was intensely interested and to which he devoted many experiments, and with plans for buildings, design of instruments, and a program of solar research. The first major investigation was on the spectrum of sunspots, and was carried on in collaboration with Adams and later with Adams and H. G. Gale in the physical laboratory. The solar equipment used was the Snow telescope and a plane-grating spectrograph with a focal length of 18 feet. The photographs were greatly superior to any obtained previously, and showed a wealth of detail, with many solar lines strengthened or weakened, and a great number of lines either not present or very faint in the solar spectrum. These latter lines were afterward identified as due to bands, those of magnesium hydride by Fowler, and those of titanium oxide and calcium hydride by the Mount Wilson observers. One result of especial interest was that in the spectrum of many elements (such, for example, as iron) some lines were greatly strengthened while others were but slightly affected, if at all. The inquiring mind of Hale determined to learn the cause of this behavior, and for this purpose had recourse to the physical laboratory.

It is difficult at the present time, long after the temperature classification of spectral lines, the development of the ionization theory, and the quantitative analysis of spectra according to energy-levels, to realize the situation that existed when these observations

of sunspot spectra were made in 1905. Owing mainly to the researches of Lockyer, the distinction between "enhanced" lines, those which first appear or are greatly strengthened in the spectrum of the spark as compared with the arc, and the normal arc lines was known and its importance recognized; but very little distinction could be made among the arc lines themselves. Experiments had shown that $\lambda$ 4227 of calcium could be obtained in the spectrum of a flame, and it was sometimes called a "low-temperature line"; but no appreciable amount of work had been carried on in this field.

Hale started with the simple and altogether natural working hypothesis that the differences of intensity between the same lines in spots and on the disk of the sun are due to the lower temperature of spots. Accordingly, an investigation was begun in the laboratory to study the effects of temperature on the spectra of various elements. The arc spectrum of iron was photographed, first with a rotating arc, then in an arc through which currents of widely varying amount could be passed, and finally in the outer flame and in the central core of an arc. The results were decisive in showing a great difference in behavior among the lines, some being relatively strong in the spectrum of the low-current arc and the cooler outer flame, while others were strong only in the hot central core and in the high-current arc. Comparison with the spectrum of sunspots at once showed that the "low-temperature" lines of the laboratory were just those which were greatly strengthened in spots while the "high-temperature" lines were but little affected. The final evidence that the effect was due to temperature was provided by a simple electric furnace, which gave results precisely similar to those found with the arc.

This investigation was of great importance not only because its results explained many of the most important phenomena of sunspots but also because it had far-reaching consequences. The classification of lines according to temperature behavior made at this time, and improved and greatly extended by King with the electric furnace, laid the foundation for much of the analysis of spectra in later years. The discovery of the weakening of the enhanced lines in sunspot spectra and the ensuing laboratory investigations led to the suggestion that low density might be favorable to the production

of enhanced lines, a conclusion so fundamental in the theory of ionization. Finally, these results had extremely wide applications to stellar spectra, aiding in determinations of temperature and density, and through successive stages leading to the spectral differentiation of giant and dwarf stars and the discovery of the spectroscopic method of determining parallaxes.

Observations with the spectroheliograph were continued regularly by Hale and Ellerman during these years at Mount Wilson, and some experimental work with this instrument led to results which culminated in Hale's most brilliant discovery. Sufficient progress had been made in sensitizing photographic plates to red light by means of dyes, especially by R. J. Wallace, to permit the use of the spectroheliograph with the $\alpha$ line of hydrogen. Photographs with this line at once showed a variety of detail, both in dark and bright flocculi, which had not been seen previously in observations with the $H\delta$ line. Most important of all, Hale's skilful examination at once detected in the curved form of the flocculi about sunspots evidence of vortical motion. Further observations fully confirmed this opinion, in one case a very long dark flocculus, which showed gradual curvature as it approached a double spot, finally forking and being drawn into the two centers of the vortex. That he foresaw at once the consequences of this discovery is shown by two quotations from his article on solar vortices:

> In view of the fact that the distribution of the hydrogen flocculi frequently resembles that of iron filings in a magnetic field, it is interesting to recall the exact correspondence between the analytical relations developed in the theory of vortices and in the theory of electromagnetism.
>
> Double lines, which look like reversals, have recently been photographed in spot spectra with the 30-foot spectrograph of the tower telescope, confirming the visual observations of Young and Mitchell. It should be determined whether the components of these double lines are circularly polarized in opposite directions, or, if not, whether other less obvious indications of a magnetic field are present. I shall attempt the necessary observations as soon as a suitable spot appears on the sun.[1]

On June 25, 1908, Hale obtained a series of photographs with the 30-foot spectrograph, using a Fresnel rhomb and Nicol prism. These

[1] *Contributions from the Mount Wilson Observatory*, No. 26; *Astrophysical Journal*, **28**, 311, 1908.

gave unmistakable evidence of the Zeeman effect and of the presence of a magnetic field in sunspots.

Following this remarkable discovery, Hale devoted considerable time to measurements of the strength of field in spots, comparisons with laboratory results, studies of plane polarization across the lines of force, and experimental work on vortex models. Since the resolution of the components of many of the spot lines was beyond the power of the 30-foot spectrograph, he decided to postpone further extensive investigations until the completion of the 150-foot tower telescope and the 75-foot spectrograph, which were under construction in 1909 and 1910. The decreasing sunspot activity, however, and the scarcity of spots led him at this time to undertake an investigation particularly well suited for sunspot minimum but extraordinarily difficult and exacting because of the smallness of the quantities involved. This was the problem of the existence of a general magnetic field of the sun as shown by the Zeeman effect. Here there could be no question of the separation of lines into components, but, as computation showed, only of minute displacements, when the Nicol prism was rotated, of three or four thousandths of a millimeter even on the great scale of the spectra with the 75-foot spectrograph. A large amount of observational material was required, and the measurements were undertaken by van Maanen and other members of the Observatory staff. Positive results were obtained which were discussed by Hale in an extensive article written in March, 1913; but the problem was rendered even more difficult by the apparent failure of certain lines to show the expected effect. The investigation was continued by Hale up to the end of his life, and he developed several ingenious and effective devices for measuring the minute quantities involved. New series of photographs were obtained near the sunspot minima of 1922 and 1932; and the spectra were measured, but with somewhat inconsistent results. It is very difficult to draw a definite conclusion regarding the outcome of this long research undertaken with such great skill and patience. The presumption for the existence of the general magnetic field seems to be strong, but the definite proof may have to await new methods or improved instruments.

In the autumn of 1918, Hale published, in collaboration with

Ellerman, Nicholson, and Joy, one of the most important of his papers, entitled "The Magnetic Polarity of Sunspots." The discovery of the reversal of polarity of sunspots with the spot cycle, fully confirmed by later results, should be ranked with that of the Zeeman effect in its importance to solar theory. In this article are also found the extremely interesting conclusions that the two members of a bipolar spot group have opposite polarities, and that the polarities of the preceding and following spots in such groups are opposite in the northern and southern solar hemispheres. Seven years later, in 1925, Hale and Nicholson, with more extensive material, discussed the law of sunspot polarity as based on 1735 groups observed between 1908 and 1925, a period covering two maxima and minima. The reversal of polarity with the spot cycle was fully established, with only 41 exceptions in the entire number observed. The full sunspot period, corresponding to the interval between the successive appearances in high latitudes of spots of the same magnetic polarity, is, therefore, twenty-three years. The reversal of polarity was found to be a sudden phenomenon unheralded by any increase of spots of irregular polarity or other exceptional event. It is fitting that a series of communications containing such brilliant discoveries in solar physics should have come to a close with one of such fundamental and far-reaching importance.

The invention of the spectrohelioscope and its numerous applications are described by Hale in a series of four publications between 1929 and 1931. The fascination which changing solar phenomena had constantly held for him from his boyhood at Kenwood was never stronger than during this last period of his active observing. In these articles he gave many results of his own observations, summarized and discussed progress in studies of the sun, and outlined a program for obtaining continuous records of solar activity through a series of stations distributed suitably in longitude throughout the world. These plans he lived to see largely accomplished. Two of the most promising fields for solar study, the relationship of solar eruptions to terrestrial phenomena and the recording of rapidly changing prominences and eruptions through continuous high-speed photography, Hale fully recognized and frequently discussed, and had health permitted, he would doubtless have developed them exten-

sively with the aid of all the powerful resources of modern instruments and technique.

Remarkable as were Hale's personal discoveries in science, his contributions in other ways were no less great. Apart from the accomplishments of the institutions he organized and the co-operative investigations he initiated, his influence through the opportunities he afforded to other scientists to carry on special investigations was of immense value. He brought E. F. Nichols to the Yerkes Observatory to make the first measurement of the heat of a star, and Michelson to Mount Wilson to measure successfully a star's diameter and improve so greatly our knowledge of the velocity of light. He was in constant touch with scientists all over the world, corresponding with them, studying their problems and methods, and often offering facilities for the extension of the work upon which they were engaged. For this reason Kapteyn came to Mount Wilson soon after the completion of the 60-inch telescope, to aid in the formulation of a program of stellar research; Julius, to investigate the effects of anomalous dispersion where observational material was immediately available; and Abbot, to undertake measurements of the solar radiation.

The great contributions made by Hale to science were recognized by learned societies all over the world, and the record of the honors and distinctions conferred upon him is a long and distinguished one. He was a member both of the leading academies and the scientific societies of England, France, Italy, Holland, Sweden, Norway, Belgium, and several other countries, in addition to those of the United States; was elected president of the International Council of Scientific Societies in 1931; and served as a member of the Committee on Intellectual Co-operation of the League of Nations in 1922. His foreign membership in the Royal Society of London dated from 1909, and his associateship in the Institut de France (Académie des Sciences) from 1919. Practically every distinction in the form of medals which scientific societies at home and abroad could award for contributions to astrophysics was bestowed upon Hale, his extensive list of honors culminating in the award of the Copley Medal of the Royal Society in 1932. As a trustee of the California Institute of Technology and of the Huntington Library and Art Gallery,

he also exercised a powerful influence upon the intellectual and cultural life of the city in which he lived; and an interesting tribute to his accomplishments in this direction was the award by the city of Pasadena of the Noble Medal for civic service.

To his own colleagues Hale was an unfailing source of help and encouragement. His insight and resourcefulness, his enthusiasm and joy in his own work and that of others, his hopefulness, modesty, and great personal charm, made any association with him a constant delight. So strong was the influence of his personality that even in the later years of his illness, when, of necessity, he had to withdraw from frequent contact with his associates, the sense of his presence and knowledge of his continuing thought of them and their work remained as a permanent influence. Finally, in its breadth of interests in the problems not only of physical science but of those of the imagination and human affairs, Hale's life was a most remarkable one. It is perhaps symbolic of this man of great gifts and wide horizons that he who had devoted his life to the study of the nearest star should find his last deep interest in an instrument destined to reveal the remotest objects of our physical universe.

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# A COMPARISON OF SPECTROSCOPIC AND TRIGONOMETRIC PARALLAXES*

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ABSTRACT

The *calibration of spectroscopic absolute magnitudes* has been made by selecting groups of stars with similar intensities for pairs of "sensitive" lines and adjusting the calibration-curves so that the absolute magnitudes which these lines give for each group shall agree with those determined for the same stars from other data (trigonometric parallaxes, proper motions, etc.). Since the spectroscopic data are affected by accidental errors, this practically unavoidable process gives a *regression-curve* which inevitably *underestimates* the deviations of the individual absolute magnitudes from the general mean for the spectral class concerned, while giving an accurate value for this mean.

The magnitude of this effect can be determined by the comparison of the spectroscopic and trigonometric parallaxes of the same stars, provided that the accidental errors of the latter are known.

*Formulae are developed* for the *determination of the constants involved* in this problem, and of the *statistical errors* of the constants derived from samples of finite size.

The spectroscopic parallaxes of *Mount Wilson Contribution* No. 511 are then compared with the trigonometric parallaxes given in *Schlesinger's General Catalogue* (Yale, 1936). For the *1140 stars of the main sequence* which are available, the *real dispersion* in absolute magnitude about the mean for the subclass of the Draper classification corresponds to a standard deviation of $\pm 34$ *per cent in the parallax*, while the *errors of the spectroscopic determination* (including the effects of real differences between stars whose sensitive lines appear the same) give a standard deviation of $\pm 38$ *per cent.* The *means* for all the stars of a given subclass are *accurate within 5 per cent*, except for a few faint M dwarfs. The differences between the individual spectroscopic absolute magnitudes and the mean for the subclass should be multiplied by 1.19 to give an improved calibration for groups selected according to the spectroscopic criteria, but by 2.6 in order to give values which will be correct in the mean for groups of stars selected by criteria independent of the spectroscopic data. When these stars are *divided into seven groups according to spectral type*, the *results for all are similar*, although with somewhat larger fluctuations than would be anticipated from errors of sampling. Comparison of the mean for groups of stars selected by spectroscopic absolute magnitude and by proper motion fully confirms the anticipated differences.

For the *giant stars*, 732 in number, the parallaxes are smaller; but good mean results can be obtained. The standard deviation in parallax, corresponding to *real dispersion in absolute magnitude*, is $\pm 52$ *per cent* (increased by the inclusion of supergiants and subgiants), while that arising from the *spectroscopic errors* is $\pm 38$ *per cent.* The *general means* of the spectroscopic and trigonometric parallaxes *agree perfectly.* The differences from the mean require no correction for grouping according to spectroscopic criteria, but a factor of 1.6 when the criteria are independent.

A special discussion of 57 stars of great absolute brightness shows that the *probable errors* assigned in "*Schlesinger's Catalogue*" *represent correctly* the true accidental errors of the determinations. *Formulae and tables are given* (sec. 13) for the rapid application of the *corrections* to the spectroscopic parallaxes. These are important for only a small fraction of the whole number of stars. Their applicability to supergiants is doubtful.

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## 1. THE CALIBRATION OF SPECTROSCOPIC ABSOLUTE MAGNITUDES

The validity of the spectroscopic method of determining absolute magnitudes and of the resulting parallaxes is fully established; but the calibration of the spectroscopic data presents certain complications. Suppose that we have, for a considerable number of stars, estimates or measures of the spectroscopic criteria associated with absolute magnitude, and also determinations of the "observed" absolute magnitudes, based on other evidence. An obvious and direct method of calibration is to divide the stars into groups, within each of which the spectroscopic criteria are the same or very similar; take the mean of the observed absolute magnitudes for each group; and plot these means against the numbers representing the mean of the spectroscopic criteria. When calibration-curves of this sort have been drawn separately for stars of different sorts, e.g., giants and dwarfs, each such class of stars presents, for our present purpose, an independent problem. In discussing this, we may assume that we are dealing with stars within a narrow range of spectral type as ordinarily defined (e.g., $dG_4$) and that their number is sufficient to justify a statistical discussion. For main-sequence stars thus selected the dispersion of the true absolute magnitudes about their mean is usually not great, the standard deviation being less than 1 mag. For a group containing ordinary giants, supergiants, and subgiants it is greater. We will here ignore complications which may arise if the criteria derived from different pairs of lines are not fully consistent, and any systematic errors which may be present in the observed absolute magnitudes, and discuss the influence of the accidental errors. Both the spectroscopic and "observed" data will contain such errors; and, since the stars have been grouped according to the values of the former, the calibration will give a regression-curve, whose course is influenced by these errors in a well-known manner.

## 2. THE REGRESSION-CURVES

Let $\overline{M}$ be the mean of the true absolute magnitudes, for all the stars in the group, and

$$M = \overline{M} + w \qquad (1)$$

the true absolute magnitude of an individual star. For the "observed" absolute magnitudes $O$ let $O_1$ be the mean systematic and $x$ the accidental error. Then

$$O = M + O_1 + x = \overline{M} + O_1 + w + x . \qquad (2)$$

If the calibration-curves were ideally drawn, we would have for the spectroscopic absolute magnitude $S$

$$S = \overline{M} + w + y ,$$

where $y$ is the accidental error. The curve actually employed may, however, require correction in zero point and slope. The former leads to a constant correction $S_1$. The latter may be allowed for by multiplying the spectroscopically determined difference $w + y$ by a factor $l$, so that

$$S = \overline{M} + S_1 + l(w + y) . \qquad (3)$$

A complete description of changes in the calibration-curve would include changes in its shape also; but these cannot be expressed by simple equations, and the summary process of correction, which amounts to changing only the zero point and horizontal scale of the curve, will be adopted here.

We must now make assumptions regarding the quantities $w$, $x$, and $y$. To assume that their mean values are zero amounts only to defining $\overline{M}$, $O_1$, and $S_1$, so that in the mean of a very large number of cases

$$\bar{O} = \overline{M} + O_1 , \qquad \bar{S} = \overline{M} + S_1 .$$

Then

$$O - \bar{O} = w + x , \qquad S - \bar{S} = l(w + y) . \qquad (4)$$

We will next assume that $w^2$, $x^2$, and $y^2$ have definite mean values $W^2$, $X^2$, and $Y^2$, the same for all large random samples of the group (with the usual statistical tolerance); that is, they are drawn from statistically homogeneous "populations."

Finally, we will assume that the mean values of the products $xw$, $xy$, and $wy$ tend toward zero for a large number of entries. For $xw$

and $xy$ this may be accepted, since by hypothesis $x$, the error of $O$, depends on observations of some kind quite independent from the rest. It is not necessary to assume that the distribution of $x$ is normal, or symmetrical about the mean; only that there is no tendency for its average value to be influenced by the observational errors of $S$, or by the fact that the star observed is absolutely brighter or fainter than the average.

Correlation between $w$ and $y$ would mean that for stars of great and small absolute brightness the effect of absolute magnitude upon the spectroscopic criteria was different. This is entirely possible; but the *mean* effect (with which we are here concerned) will be automatically allowed for in drawing a good calibration-curve, and errors in this process have already been taken into account (approximately) in the factor $l$. The assumption that $wy = 0$ is therefore a legitimate approximation.

If then we set

$$u = S - \bar{S} = l(w + y) , \qquad u' = O - \bar{O} = w + x , \qquad (5)$$

we have the mean values

$$\left.\begin{aligned} \overline{u^2} &= l^2W^2 + l^2Y^2 , \\ \overline{u'^2} &= W^2 + X^2 , \\ \overline{uu'} &= lW^2 . \end{aligned}\right\} \qquad (6)$$

We may now define the regression line representing the mean values of $O$ for specified values of $S$, by solving by least squares the equations of condition

$$au = u' ,$$

where $a$ is the constant to be determined. The solution (in the Gaussian notation) is

$$a[u^2] = [uu'] ; \qquad (7)$$

whence, by (6), the regression line is

$$u' = au = \frac{W^2}{l(W^2 + Y^2)}\, u . \qquad (8)$$

This process ignores the possible curvature of the regression-curve; but, since we are supposed to start with an approximately calibrated set of spectroscopic absolute magnitudes, this should not be serious.

To secure the best agreement between the mean observed and spectroscopic absolute magnitudes, for stars *grouped according to the values of the latter*, we must multiply the provisional values of $S - S_0$ with which we started by the constant $a$ defined by (7). With these newly calibrated values, the regression line will be $u = u'$ and the new value of $l$ will be

$$l_1 = \frac{W^2}{W^2 + Y^2} . \tag{9}$$

This method of calibration involves the well-known tendency to select positive errors when $S$ is large, and negative errors when it is small.

For groups selected by some impartial method, such that there is no preference for positive or negative errors in either $S$ or $O$, the relation is different. To obtain significant results we must have a criterion which separates—partially, at least—stars of different real brightness. For such a group the means of $x$ and $y$ will tend toward zero, but that of $w$ toward some value $w_n$, so that

$$\bar{u}_n = l w_n ; \qquad \overline{u'_n} = w_n .$$

The mean points will lie on the line

$$\overline{u'_n} = \frac{\bar{u}_n}{l} . \tag{10}$$

Finally, for groups selected according to the value of $O$, we find, for the equation of the regression line,

$$u' = \frac{W^2 + X^2}{l W^2} u . \tag{11}$$

There is nothing new in an analysis of this type.[2] It should, however, be noted that it has not been assumed at any point that the

[2] Cf. Seares, *M.N.*, **84**, 15–20, 1923.

distribution of the quantities $w$, $x$, and $y$ follows the normal error law. The assumptions actually made are much less restrictive. If, however, the distributions of $w$, $x$, and $y$ are normal, it is easy to show that the distribution of $u$ and $u'$ follows the normal law in two variates, so that the regression-curves are actually the straight lines (8) and (11).

The equations (6) do not suffice to determine the unknowns. But, if the dispersion $X$ can be calculated from a study of the "observed" data, $W$, $Y$, and $l$ may be found. It should be noted, however, that the regression line (8) involves only $[uu]$ and $[uu']$ and is independent of the value of $X$ (as it obviously must be, since, when taking means of $O$ for a fixed value of $S$, the errors of $O$ tend to vanish in the mean of a large number of cases).

### 3. APPARENT AND REAL DISPERSIONS

If we should adjust our calibration so that the slope of the regression line (8) was 1, that of the "impartial" line (10) would be

$$1 + \frac{Y^2}{W^2}.$$

This slope will be steeper, the greater is the ratio of the mean accidental error of a spectroscopic determination of absolute magnitude to the true standard deviation of the absolute magnitudes about their mean.

The standard deviation of the spectroscopic absolute magnitudes, as thus calibrated, would be

$$\sigma_s^2 = \overline{u^2} = \frac{W^4}{W^2 + Y^2}.$$

The true standard deviation, including the effects of accidental errors, is

$$\sigma^2 = W^2 + Y^2.$$

Hence, with this adjustment,

$$\sigma_s = \frac{\sigma W^2}{W^2 + Y^2}.$$

If the calibration is not thus adjusted,

$$\sigma_s = l(W^2 + Y^2)^{\frac{1}{2}} = l\sigma .$$

The method of calibration leads therefore to an underestimate of the errors of the spectroscopic parallaxes and to the real dispersion of the absolute magnitudes. This underestimate may be serious, but it is in no sense an error of calculation; it is the inevitable result of the method of calibration, and its amount is not diminished by increasing the number of stars, so long as the accidental errors and true dispersion remain the same. This type of calibration appears, nevertheless, to be the best that is generally available; for the correction from the regression to the "impartial" line can be made only when the accidental errors of the individual "observed" values can be accurately estimated, which is not always possible, especially when mean parallaxes, etc., have been found from proper motions. It should be emphasized that the underestimate affects only the deviations from the mean for the group (in both directions). The group-means themselves are undisturbed.

### 4. CORRESPONDING RELATIONS FOR THE PARALLAX

The practical determination of these quantities is complicated by the fact that there is no direct method for determining absolute magnitude. Some of the best methods for getting mean values, such as those depending on proper motions, provide no way for finding the errors $u'$ for individual stars. The mean value of $u'^2$ may be well determinable in other ways, but those of the products $uu'$ cannot be found at all.

For an investigation of the correlations and regressions, the best material is offered by trigonometric parallaxes. It is useless to compute absolute magnitudes directly from these, since some of them are negative. We must transform the spectroscopic absolute magnitudes into quantities of the dimensions of a parallax.

The actual spectroscopic parallaxes, $s$, are inconvenient for this purpose, since they are affected by the very large dispersion in the distances of the stars. We may obtain convenient quantities by com-

puting the values $s'$ which the spectroscopic parallaxes would have if the stars all had an apparent magnitude equal to $S_0$. Then[3]

$$S = S_0 + 5 + 5 \log s' ,$$

and, by (5),

$$u = S_0 - \bar{S} + 5 + 5 \log s' ;$$

whence

$$\begin{aligned} 10s' &= \exp 0.46\ (u + \bar{S} - S_0) \\ &= [\exp 0.46(\bar{S} - S_0)]\ (1 + 0.46u + 0.106u^2 \\ &\quad + 0.016u^3 + 0.002u^4 \ldots . ) . \end{aligned}$$

The mean value of $s'$ is given by

$$10\overline{s'} = [\exp 0.46\ (\bar{S} - S_0)](1 + 0.106\overline{u^2}) , \tag{12}$$

the higher terms being negligible.

If we set

$$s' = \overline{s'_0} + v ,$$

we have

$$\frac{v}{\overline{s'_0}} = 0.46\ \frac{u + 0.23\ (u^2 - \overline{u^2}) \ldots .}{1 + 0.106\overline{u^2}} . \tag{13}$$

In practice $S_0$ has been taken nearly equal to $\bar{S}$, so that $\overline{s'}$ is close to $0''.10$.

If the dispersion in $u$ is not too great, $u$ and $v$ will be nearly proportional over most of their range. If the distribution of $u$ is normal, that of $v$ will be asymmetrical; but, in the absence of a priori knowledge of the distribution of absolute magnitudes, consideration of this would be a refinement appropriate only to a second approximation. If $m$ is the apparent magnitude and we set

$$\left.\begin{aligned} s' &= fs , \\ \log f &= 0.2(m - S_0) . \end{aligned}\right\} \tag{14}$$

we have

[3] In this paper "log" implies the base 10; "*ln*," the base *e*.

For comparison with the "reduced" spectroscopic parallaxes $s'$, we must multiply the trigonometric parallaxes $t$ by the factors $f$, obtaining

$$t' = ft .$$

If $p$ is the true parallax, we may set

$$t = Cp(1 + z) , \tag{15}$$

where $C$ is a constant factor representing the mean effect of systematic error and $z$ represents the accidental error of observation.

If $M$ is the true absolute magnitude,

$$m - M = -5 - 5 \log p ;$$

whence, by (1) and (14),

$$\log f = -1 - \log p + 0.2 (\overline{M} - S_0 + w) .$$

Hence,

$$\begin{aligned} 10t' &= C (1 + z) \exp 0.46 (\overline{M} - S_0 + w) \\ &= C [\exp 0.46 (\overline{M} - S_0)](1 + z + 0.46w + 0.106w^2 \\ &\qquad + 0.46zw + \dots .) . \end{aligned}$$

The mean value is

$$10\overline{t'} = C [\exp 0.46 (\overline{M} - S_0)](1 + 0.106\overline{w^2} + \dots .) .$$

Setting

$$t' = \overline{t'} + v' ,$$

we have

$$\frac{v'}{\overline{t'}} = \frac{z + 0.46w + 0.106 (w^2 - \overline{w^2}) \dots .}{1 + 0.106\overline{w^2}} . \tag{16}$$

If $x$ is the change in absolute magnitude corresponding to the change $z$ in parallax,

$$1 + z = \exp 0.46x ,$$

or

$$z = 0.46x + 0.106x^2 \ldots . \quad .$$

Hence, with sufficient approximation for our purpose, we have

$$\left.\begin{aligned} \frac{v'}{l'} &= 0.46\,(w + x) + \ldots . , \\ \frac{v}{s'} &= 0.46\,l(w + y) + \ldots . \end{aligned}\right\} \tag{17}$$

In an exhaustive investigation, which sought to determine the form of the distribution functions of the errors, and of the regression-curves, it might be necessary to take account of the terms neglected in (17); but the present study represents a first, and not a final, approximation.

If

$$w_1 = 0.46w , \qquad W_1 = 0.46W , \textit{ etc.},$$

we now have in place of (6)

$$\left.\begin{aligned} \frac{\overline{v^2}}{\overline{s'^2}} &= l^2(W_1^2 + Y_1^2) , \\ \frac{\overline{v'^2}}{\overline{l'^2}} &= W_1^2 + X_1^2 , \\ \frac{\overline{vv'}}{\overline{s'}\,\overline{l'}} &= l\,W_1^2 . \end{aligned}\right\} \tag{18}$$

The quantities $v$ and $v'$ may be found from the observed spectroscopic and trigonometric parallaxes. Moreover, the probable errors of the latter, if reliable, determine $X_1^2$, so that the other unknowns may be evaluated.

## 5. WEIGHTS

In applying these principles we must take account of the great differences in the percentage errors of the trigonometric parallaxes. If $r$ is the probable error assigned to $l$, that of the "reduced" parallax $l'$ will be $r' = fr$. The weight $p_l$ of $l'$ varies inversely as $r'^2$.

In a discussion of the masses of the stars, already in progress, the unit of weight has been taken to correspond to a probable error of $\pm 28$ per cent of the mean parallax, and this convention has been adopted here. The mean value $\overline{l'}$ is usually close to $0''.10$, so that we may adopt

$$p_l = \left(\frac{0''.028}{r'}\right)^2 . \qquad (19)$$

The percentage error of a spectroscopic parallax is approximately constant. Its value was provisionally assumed to be $\pm 20$ per cent,

TABLE 1

| $r'$ | Exact Weight | Assigned Weight | $r'$ | Exact Weight | Assigned Weight |
|---|---|---|---|---|---|
| $0''.000$ | 2.00 | 1.8 | $0''.0555$ | 0.23 | 0.20 |
| .0085 | 1.69 | 1.4 | .0655 | .17 | .14 |
| .0165 | 1.20 | 1.0 | .0785 | .122 | .10 |
| .0235 | 0.84 | 0.7 | .0945 | .086 | .07 |
| .0305 | 0.60 | 0.5 | .1125 | .061 | .05 |
| .0385 | 0.42 | 0.35 | .1345 | .043 | .035 |
| .0485 | 0.29 | 0.25 | .1625 | .030 | 0.025 |
| 0.0555 | 0.23 | | 0.1865 | 0.022 | |

which gives $p_s = 2$ in all cases. Only a large error in this assumption would change the weights enough to require revision.

To assign different weights to the same star when discussing $s'$ and $t'$ has obvious disadvantages. Both quantities, for any star, have therefore been given a weight $p$, defined by the usual equation

$$\frac{1}{p} = \frac{1}{p_s} + \frac{1}{p_t} . \qquad (20)$$

To save labor in the computation, uniform weights were assigned to all stars for which $r'$ lay between certain limits, as in Table 1. A coarser subdivision would have been practically satisfactory.

## 6. FINAL EQUATIONS

We now have

$$\frac{N}{N-1}\frac{[pv'v']}{\overline{l'^2}} = [pw_1^2] + 2[pw_1x_1] + [px_1^2] .$$

The factor $N/(N-1)$ is introduced to take account of the fact that the number $N$ of entries is moderate, instead of very large, as previously supposed.

The mean value of $w_1^2$ is always $W_1^2$, while, for an entry of weight $p_t$, that of $x_1^2$ is $X_1^2/p_t$, where $X_1$ is the standard deviation for unit weight.

Hence (since $p/p_t = 1-p/p_s$ and $p_s$ is constant),

$$\frac{N}{N-1}\frac{[pv'v']}{\overline{l'^2}} = [p]W_1^2 + \left[\frac{p}{p_t}\right] X_1^2 = [p]W_1^2 + \left\{N - \frac{[p]}{p_s}\right\} X_1^2 . \quad (21)$$

To $p_t = 1$ corresponds a probable error $r' = \pm 0.028$, and a mean error $X_0 = \pm 0.0416$ for $v'$. Hence

$$X_1 = \frac{X_0}{\overline{l'}} , \quad (22)$$

and we have

$$[p]W_1^2 = \frac{1}{\overline{l'^2}}\left\{\frac{N}{N-1}[pv'v'] - \left(N - \frac{[p]}{p_s}\right) X_0^2\right\} .$$

If we set

$$\frac{N}{[p]} - \frac{1}{p_s} = B\overline{l'^2} , \quad (23)$$

we have

$$[p]W_1^2 + [p]BX_0^2 = \frac{N}{N-1}\frac{[pv'v']}{\overline{l'^2}} = NE' . \quad (24)$$

Similarly, we find

$$[p]lW_1^2 = \frac{N}{N-1}\frac{[pvv']}{\bar{s}'\bar{l}} = NF\,, \tag{25}$$

$$[p]l^2(W_1^2 + Y_0^2) = \frac{N}{N-1}\frac{[pvv]}{\bar{s}'^2} = NE\,, \tag{26}$$

where $Y_0 = Y_1/\sqrt{p_s}$ and is the mean accidental error of a spectroscopic determination of parallax.

Since $p_s = 2$, and $X_0 = 0.0416$, these equations determine $W_1$, $l$, and $Y_0$.

When several groups are to be combined with one solution, these equations may be added as they stand, introducing the proper numerical values of $B$ and $X_1$ into each. If $\bar{l}'$ is the same for all the groups, equations (22) and (23) may be used with the combined values of $N$ and $[p]$; otherwise this process is slightly inaccurate.

The group-means $\bar{s}'$ and $\bar{l}'$ will themselves be affected by accidental errors. Better values, $s_0 l_0$, may be obtained by plotting $\bar{s}'$ and $\bar{l}'$ against the spectral class and drawing smooth curves. If

$$\Delta s = s_0 - \bar{s}'\,, \qquad \Delta t = t_0 - \bar{l}'\,,$$

we have, referred to the new standard:

$$\left.\begin{aligned} [pvv]_0 &= [pvv] + [p]\Delta s^2\,, \\ [pv'v']_0 &= [pv'v'] + [p]\Delta t^2\,, \\ [pvv']_0 &= [pvv'] + [p]\Delta s\Delta t\,. \end{aligned}\right\} \tag{27}$$

To retain the factors $N/(N-1)$ now slightly exaggerates the correction for the finite size of the groups; but this is practically harmless.

### 7. STATISTICAL ERRORS OF SAMPLING

With groups of moderate size the quantities $[pvv]$, etc., will be subject to considerable fluctuations, owing to random sampling—of the order of $N^{-1/2}$ times their mean values.

If we set

$$e = l(w + y)\,, \qquad e' = w + x\,, \tag{28}$$

we have, by (24), on the average,

$$[pee] = \frac{N}{N-1}\frac{[pvv]}{s_0^2} = NE \text{ or } \overline{pee} = E\,,$$

where the bar denotes the mean value.

For an individual group of $N$ data, this will not be exactly true; and if we set

$$[p] = NP\,, \qquad N - \frac{[p]}{p_s} = NQ\,, \tag{29}$$

we may write

$$\left.\begin{aligned} &\overline{pee} = E + g \text{ or } Ng = [pee] - NE = [pee] \\ &\qquad\qquad - NPl^2(W_1^2 + Y_0^2)\,, \\ &\text{and similarly, from (21) and (26),} \\ &\overline{pe'e'} = E' + g'\,, \qquad Ng' = [pe'e'] - NE' \\ &\qquad\qquad = [pe'e'] - N(PW_1^2 + QX_1^2)\,, \\ &\overline{pee'} = F + h\,, \qquad Nh = [pee'] - NF = [pee'] - NPlW_1^2\,. \end{aligned}\right\} \tag{30}$$

For the mean-square values we have

$$N^2\overline{g'^2} = \overline{[pe'e']^2} - 2NE\overline{[pe'e']} + N^2E'^2 = \overline{[pe'e']^2} - N^2E'^2\,, \text{ and so on;}$$

also for the product

$$N^2\overline{gh} = \overline{[pee][pee']} - N^2EF\,, \text{ etc.}$$

We have here to take the means of a large number of squared sums like $[pee]$ with different values of $w$, $x$, and $y$, but the same weights $p$ throughout. We have

$$[pe'e']^2 = \{[px^2] + 2[pxw] + [pw^2]\}^2\,.$$

The mean values of all sums containing odd powers of $x$ or $w$ vanish, so that

$$\overline{[pe'e']^2} = \overline{[px^2]^2} + \overline{[pw^2]^2} + 4\overline{[pxw]^2} + 2\overline{[px^2][pw^2]}\,.$$

Now

$$[px^2]^2 = (p_1x_1^2 + p_2x_2^2 \ldots .)^2 = p_1^2x_1^4 + p_2^2x_2^4 + 2p_1p_2x_1^2x_2^2 \ldots . .$$

For any entry (say $x_2$) there is a trigonometric weight $p_{t_2}$, fixed for all the successive trials, such that

$$\overline{x_2^2} = \frac{X_1^2}{p_{t_2}}.$$

For $x^4$ we may assume

$$\overline{x^4} = (1 + c)\,(\overline{x^2})^2\,. \tag{31}$$

For a normal distribution, $c = 2$. We will at least assume it constant for all entries. We then find

$$\overline{[px^2]^2} = X_1^4 c \left[\frac{p^2}{p_t^2}\right] + X_1^4 \left[\frac{p}{p_t}\right]^2.$$

Now $p/p_t = 1 - p/p_s$, and $p_s$ is constant. Hence,

$$\left.\begin{aligned} \left[\frac{p}{p_t}\right] &= N - \left[\frac{p}{p_s}\right] = NQ\,, \\ \left[\frac{p^2}{p_t^2}\right] &= N - 2\left[\frac{p}{p_s}\right] + \frac{[p^2]}{p_s^2}\,. \end{aligned}\right\} \tag{32}$$

Set

$$\overline{p^2} = (1 + A)(\overline{p})^2 = (1 + A)P^2 \tag{33}$$

and we have

$$\left[\frac{p^2}{p_t^2}\right] = N(Q^2 + AQ'^2)\,, \tag{34}$$

where

$$Q' = 1 - Q\,.$$

We find also

$$\begin{bmatrix} p^2 \\ p_l \end{bmatrix} = NP(Q - AQ') \,. \tag{35}$$

We now have

$$\left.\begin{aligned} \overline{[px^2]^2} &= NcX_1^4(Q^2 + AQ'^2) + N^2Q^2X_1^4 \,. \\ \text{Similarly, but more easily, we find} \\ \overline{[py^2]^2} &= Nc(1 + A)P^2Y_0^4 + N^2P^2Y_0^4 \,, \\ \overline{[pw^2]^2} &= Nc(1 + A)P^2W_1^4 + N^2P^2W_1^4 \,, \\ \overline{[pxy]^2} &= NP(Q - AQ')X_1^2Y_0^2 \,, \\ \overline{[pxw]^2} &= NP(Q - AQ')X_1^2W_1^2 \,, \\ \overline{[pwy]^2} &= NP^2(1 + A)W_1^2Y_0^2 \,, \\ \overline{[px^2]} &= NQX_1^2 : \overline{[py^2]} = NPY_0^2 : \overline{[pw^2]} = NPW_1^2 \,. \end{aligned}\right\} \tag{36}$$

We now find, easily (setting $c = 2$),

$$\left.\begin{aligned} N\,\overline{g^2} &= 2(1 + A)E^2 \,, \\ N\,\overline{g'^2} &= 2\{E'^2 + A(E' - X_1^2)^2\} \,, \\ N\,\overline{h^2} &= F^2 + EE' + A\{F^2 + E(E' - X_1^2)\} \,, \\ N\,\overline{gg'} &= 2(1 + A)F^2 \,, \\ N\,\overline{gh} &= 2(1 + A)EF \,, \\ N\,\overline{g'h} &= 2E'F' + 2AF(E' - X_1^2) \,. \end{aligned}\right\} \tag{37}$$

The statistical mean errors of $E$, $E'$, and $F$ are, respectively, $g$, $g'$, and $h$ Under the assumed conditions $QX_1^2$ $(=QX_0^2/l_0^2)$ is constant, since $Q$ depends on the weights, which are fixed, as is $l_0$, while $X_0$ has been accurately determined from a great number of observations, largely extraneous to the present discussion. The mean error of the calculated $PW_1^2$ is therefore $g'$.

We are especially interested in the errors of the coefficients, $a$ and $1/l$, which define the calibrating lines (8) and (10). We have

$$a = \frac{\overline{pee'}}{\overline{pee}} = \frac{F + h}{E + g} \,.$$

If $a_0$ is the mean value $F/E$, we have, to the first order,

$$a - a_0 = \frac{Eh - Fg}{E^2} + \dots ;$$

whence, by (37),

$$NE^2\overline{(a - a_0)^2} = EE' - F^2 + A(EE' - EX_1^2 - F^2) . \qquad (38)$$

Similarly, since

$$\frac{1}{l} = E' - \frac{QX_1^2}{F} ,$$

we find, after some reckoning,

$$NF^4\overline{\left(\frac{1}{l} - \frac{1}{l_0}\right)^2} = (E' - QX_1^2)^2\{EE' - F^2 + A(EE' - EX_1^2 - F^2)\} + 2X_1^4F^2(Q^2 + AQ'^2) . \qquad (39)$$

Apart from the last term in (39) the ratio of the mean errors is

$$\frac{E(E' - QX_1^2)}{F^2} = \frac{W_1^2 + Y_0^2}{W_1^2} ,$$

which is the ratio of the slopes of the lines. The additional term shows that the "impartial" constant $1/l$ is always more poorly determined than the regression constant $a$.

## 8. THE OBSERVATIONS

Significant results in a statistical inquiry of this sort can be obtained only if extensive and homogeneous observational data are available. Schlesinger's *General Catalogue of Stellar Parallaxes*[4] provides all that could be desired for the trigonometric parallaxes. The systematic errors for the various observatories have been very carefully discussed and applied, and the absolute value of the corrections checked by all available criteria. Schlesinger remarks (p. iv): "I regard it as more probable than not that the correction needed to make this system absolutely true as a whole does not exceed $''.001$." The assigned probable errors have also been very carefully

[4] Yale University Observatory, 1935.

adjusted, removing the fluctuations due to random sampling, and allowing for the nonsystematic discordances between the results of different observatories. New evidence will be presented below that they correspond closely to the true accidental errors. The trigonometric parallaxes of this *Catalogue* have been adopted without modification in all cases.

The spectroscopic parallaxes here discussed are those of the latest Mount Wilson list.[5] While all these have been derived by the same method, many different calibration-curves have been employed, depending on various pairs of lines.

We are indebted to Dr. Adams for full details of these curves. One set was used for stars belonging to the main sequence and a quite independent set for giants, including supergiants and subgiants. The range of spectral class over which a given pair is sensitive to luminosity varies widely; and, within this range, several roughly parallel curves were usually employed for different spectral subclasses. The exact spectral criteria employed therefore vary from subclass to subclass (and to a smaller degree from star to star); but over a moderate spectral range this change is unimportant.

Few of the calibration-curves are rectilinear, but their curvature is smooth, and there are no examples of sharp bends of such a character (e.g., in the interval between giants and supergiants) as to make it probable that the main part of a necessary revision could not be made by changing the zero point and slope, as described in section 1.

The conditions laid down in section 1 appear therefore to be satisfied, so long as giants and main-sequence stars are kept separate and all giants are discussed together. A few exceptional objects (such as the abnormally faint A stars called "white dwarfs" in the notes), which were calibrated by special methods, have been excluded from the discussion. *Every star calibrated by the ordinary methods, however discordant, has been included.*

The published values of the spectroscopic absolute magnitudes have in all cases been adopted. The apparent visual magnitudes given in *Mount Wilson Contribution* No. 511 have been altered in a

[5] Adams, Joy, Humason, and Brayton, "The Spectroscopic Absolute Magnitudes and Parallaxes of 4179 Stars," *Mt. W. Contr.*, No. 511; *Ap. J.*, **81**, 187, 1935.

few cases—with corresponding changes in the spectroscopic parallaxes. Most of these are for double stars of wide separation for which the magnitudes given in the *Henry Draper Catalogue* refer to the combined light, but had been assumed, by the compilers of the list, to belong to the brighter component. A number of such cases were cleared up by reference to other Harvard publications.[6] For two or three stars photometric measures made at Harvard, and published in the senior author's book on parallaxes,[7] have been adopted in place

TABLE 2

STANDARD SPECTROSCOPIC MAGNITUDES, $S_0$

| MAIN SEQUENCE | | | | | | | | GIANTS | |
|---|---|---|---|---|---|---|---|---|---|
| Sp | $S_0$ | Sp | $S_0$ | Sp | $S_0$ | Sp | $S_0$ | Sp | $S_0$ |
| A1 | +0.9 | F2 | +2.8 | G3 | +4.4 | K4 | + 6.4 | G0–G5 | +0.7 |
| A2 | 1.1 | F3 | 3.0 | G4 | 4.5 | K5 | 6.7 | G6–G9 | +0.6 |
| A3 | 1.3 | F4 | 3.1 | G5 | 4.6 | K6 | 7.3 | K0 | +0.5 |
| A4 | 1.5 | F5 | 3.2 | G6 | 4.8 | M0 | 8.3 | K1–K2 | +0.4 |
| A5 | 1.7 | F6 | 3.4 | G7 | 5.0 | M1 | 8.8 | K3 | +0.3 |
| A6 | 1.9 | F7 | 3.5 | G8 | 5.1 | M2 | 9.4 | K4 | +0.2 |
| A7 | 2.0 | F8 | 3.6 | G9 | 5.3 | M3 | 10.0 | K5 | +0.1 |
| A8 | 2.2 | F9 | 3.8 | K0 | 5.5 | M4 | 10.6 | M0–M2 | −0.2 |
| A9 | 2.3 | G0 | 4.0 | K1 | 5.7 | M5 | +11.9 | M3 | −0.3 |
| F0 | 2.5 | G1 | 4.1 | K2 | 5.9 | ..... | ..... | M4–M7 | −0.4 |
| F1 | +2.7 | G2 | +4.2 | K3 | +6.2 | ..... | ..... | ..... | ..... |

of the corrected *Bonner Durchmusterung* magnitudes given in the *Henry Draper Catalogue*. One misprint in *Mount Wilson Contribution* No. 511 was discovered: for the star 4386A, $\alpha$ $17^h12^m1$, $\delta$ $-34°$ 53′, $s$ should be 0″.166 instead of 0″.116.

## 9. SUMMARY OF THE DATA

In applying the equations (14) it is not necessary to start with an accurate value of the assumed absolute magnitude $S_0$, since any error in this is automatically eliminated from $v/s'$ or $v'/t'$; but it is convenient to do so. The mean values of $S$ for each spectral subdivision were plotted, and smooth curves drawn to represent them, with the results in Table 2. The means for the giants are derived from normal giants alone.

[6] *Harvard Ann.*, **56**, 231 (No. 7), 1912; **64**, 165 (No. 6), 1912.

[7] *Pub. Carnegie Institution of Washington*, No. 147, 1911.

For each separate spectral subdivision, the values of $s'$, $t'$, $r'$, and the weight $p$ were computed for each star; the weighted means $\overline{s'}$ and $\overline{t'}$ were taken, and the residuals $v$ and $v'$ from these means tabulated.

For the main sequence the values of $\overline{s'}$ gave a smooth plot against spectral class, while those of $\overline{t'}$ were sometimes ragged. Smooth curves were drawn to represent each, and improved mean values $s_0$ and $t_0$ read from these. The quantities $[pvv]$, etc., were corrected by (27) to give the sums $[pvv]_0$ corresponding to these new means. From these the quantities $[p]B$, $NE$, $NE'$, and $NF$ defined in equations (23) to (26) were calculated. The resulting data are given in Table 3. The second column gives the number of stars, the third, the sum of their weights.

## 10. PRELIMINARY DISCUSSION

The mean weight increases along the sequence because the later-type stars have, on the average, larger trigonometric parallaxes and the percentage errors of these are smaller. The values of $\overline{s'}$ show a narrow range (except for A1 and A2, where the course of the curve adopted for $S_0$ is uncertain). The weighted mean of all the values is 0.1032, for 1140 stars.

The difference between this and the expected value $0''.100$ arises mainly from the fact that the values of $S_0$, used in calculating $s'$, represent the means of the absolute magnitudes. If $S$ is the individual absolute magnitude

$$10s' = \exp 0.46\,(S - S_0)\ ;$$

whence

$$10\overline{s'} = 1 + 0.106\,\overline{(S - S_0)^2}\,. \qquad (40)$$

From a tabulation furnished by Mr. Joy, it is found that, for 1707 main-sequence stars of spectra A0 to K3, $\overline{(S - S_0)^2} = 0.208$, so that we might expect a mean value $\overline{s'} = 0.1022$. The remaining discrepancy of 1 per cent (if real) would indicate that the stars observed trigonometrically were, on the average, 0.02 mag. fainter than the general run of those observed spectroscopically—a wholly insignificant difference.

The tendency of $\bar{s}'$ to run higher in the region F5–G5 than in G6–K6 may be real, but could be removed by altering $S_0$ by 0.1 mag. at

TABLE 3

OBSERVATIONAL DATA: MAIN SEQUENCE

| Sp | $N$ | $[p]$ | $\bar{s}'$ | $\bar{t}'$ | $R$ | $s_0$ | $t_0$ | $\frac{[p]B}{100}$ | $NE$ | $NE'$ | $NF$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A1 | 6 | 5.3 | 0".157 | 0".116 | 0.98 | 0".130 | 0".116 | 2.4 | 0.92 | 2.66 | 0.79 |
| A2 | 13 | 14.2 | .118 | .109 | 0.98 | .122 | .111 | 4.8 | 0.62 | 0.81 | −0.03 |
| A3 | 16 | 9.3 | .110 | .111 | 0.98 | .114 | .106 | 10.1 | 0.53 | 1.66 | 0.33 |
| A4 | 22 | 11.8 | .109 | .087 | 0.98 | .106 | .102 | 15.5 | 0.57 | 3.25 | 0.77 |
| A5 | 16 | 7.3 | .096 | .083 | 0.97 | .098 | .099 | 12.6 | 0.32 | 2.98 | 0.63 |
| A6 | 15 | 10.8 | .086 | .100 | 0.97 | .093 | .097 | 10.2 | 0.16 | 1.46 | 0.14 |
| A7 | 13 | 11.0 | .089 | .095 | 0.96 | .092 | .095 | 8.3 | 0.40 | 3.38 | 0.48 |
| A8 | 11 | 8.5 | .096 | .099 | 0.96 | .095 | .094 | 7.7 | 0.20 | 3.40 | 0.00 |
| A9 | 12 | 6.4 | .091 | .098 | 0.95 | .098 | .093 | 10.2 | 0.38 | 1.55 | 0.18 |
| F0 | 19 | 14.5 | .100 | .087 | 0.94 | .100 | .093 | 13.6 | 0.71 | 6.74 | 1.62 |
| F1 | 34 | 19.3 | .109 | .091 | 0.93 | .102 | .094 | 27.6 | 1.28 | 7.98 | 1.49 |
| F2 | 34 | 30.6 | .100 | .092 | 0.93 | .103 | .096 | 20.4 | 1.36 | 5.79 | 1.68 |
| F3 | 49 | 39.5 | .104 | .095 | 0.94 | .104 | .098 | 30.5 | 2.55 | 11.65 | 3.77 |
| F4 | 51 | 37.9 | .107 | .109 | 0.95 | .105 | .102 | 30.8 | 1.64 | 8.88 | 2.10 |
| F5 | 60 | 48.0 | .105 | .097 | 0.98 | .106 | .105 | 32.7 | 2.22 | 12.73 | 2.62 |
| F6 | 59 | 49.3 | .102 | .105 | 1.00 | .106 | .109 | 28.9 | 1.05 | 10.84 | 1.59 |
| F7 | 35 | 25.7 | .105 | .115 | 1.02 | .107 | .113 | 17.4 | 0.88 | 6.88 | 0.91 |
| F8 | 46 | 35.3 | .100 | .105 | 1.04 | .107 | .116 | 21.0 | 1.78 | 12.48 | 2.94 |
| F9 | 27 | 20.2 | .105 | .125 | 1.05 | .108 | .118 | 12.2 | 0.91 | 5.72 | 0.72 |
| G0 | 41 | 37.6 | .109 | .118 | 1.04 | .108 | .117 | 16.2 | 0.92 | 8.15 | 1.68 |
| G1 | 39 | 29.7 | .109 | .122 | 1.04 | .108 | .116 | 18.0 | 0.69 | 6.04 | 1.13 |
| G2 | 44 | 32.7 | .106 | .109 | 1.03 | .108 | .114 | 21.3 | 0.64 | 7.42 | 0.82 |
| G3 | 35 | 28.3 | .108 | .093 | 1.03 | .108 | .112 | 16.6 | 0.44 | 5.64 | 0.32 |
| G4 | 27 | 21.6 | .118 | .117 | 1.03 | .107 | .110 | 13.4 | 1.39 | 5.57 | 1.84 |
| G5 | 31 | 22.7 | .113 | .118 | 1.02 | .105 | .108 | 16.8 | 2.50 | 9.27 | 2.89 |
| G6 | 27 | 22.4 | .100 | .097 | 1.01 | .103 | .106 | 14.1 | 0.41 | 5.19 | 0.80 |
| G7 | 26 | 26.3 | .097 | .111 | 1.00 | .101 | .105 | 11.6 | 1.04 | 6.16 | 1.79 |
| G8 | 21 | 19.4 | .103 | .111 | 1.00 | .100 | .103 | 10.6 | 1.01 | 5.38 | 1.34 |
| G9 | 15 | 16.7 | .096 | .072 | 0.99 | .100 | .102 | 6.4 | 0.89 | 4.41 | 1.20 |
| K0 | 27 | 30.3 | .097 | .103 | 0.99 | .100 | .100 | 11.8 | 0.74 | 6.60 | 0.96 |
| K1 | 31 | 33.4 | .102 | .109 | 0.98 | .100 | .099 | 14.6 | 0.71 | 4.52 | 0.31 |
| K2 | 23 | 24.8 | .094 | .092 | 0.97 | .100 | .097 | 11.3 | 0.33 | 4.50 | 0.32 |
| K3 | 12 | 13.2 | .105 | .101 | 0.96 | .100 | .096 | 5.9 | 0.28 | 1.04 | 0.07 |
| K4 | 16 | 18.4 | .104 | .086 | 0.95 | .100 | .094 | 7.7 | 0.21 | 1.33 | 0.04 |
| K5 | 40 | 53.0 | .097 | .096 | 0.95 | .105 | .093 | 15.6 | 1.07 | 7.57 | 1.75 |
| K6 | 47 | 58.7 | .098 | .091 | 0.96 | .100 | .092 | 20.9 | 1.82 | 7.82 | 1.65 |
| M0 | 33 | 48.6 | .103 | .099 | 0.97 | .103 | .099 | 8.9 | 1.30 | 4.26 | 1.26 |
| M1 | 26 | 40.1 | .101 | .100 | 1.00 | .101 | .100 | 6.0 | 0.88 | 5.09 | 0.85 |
| M2 | 18 | 30.0 | .107 | .108 | 1.03 | .107 | .108 | 2.6 | 2.35 | 2.15 | 1.65 |
| M3 | 12 | 20.8 | .102 | .119 | 1.07 | .102 | .119 | 1.1 | 0.80 | 2.69 | 1.07 |
| M4 | 4 | 7.2 | .099 | .103 | 1.15 | .099 | .103 | 0.4 | 0.34 | 0.32 | 0.34 |
| M5 | 7 | 12.6 | 0.106 | 0.129 | ..... | 0.106 | 0.129 | 0.4 | 1.14 | 1.42 | 1.08 |

the most. The values of $\bar{t}'$ are more ragged, on account of the errors of the trigonometric parallaxes. They run definitely below 0".100

near Fo, and high between F5 and G5. The smoothed values $s_0$ and $t_0$ agree with $\overline{s'}$ and $\overline{t'}$ within the obvious scatter of the data.

For the M dwarfs the individual trigonometric parallaxes are of high percentage accuracy and the means have not been smoothed.

The correction to the assumed spectroscopic absolute magnitude which is indicated by the trigonometric parallax is $\Delta M = 5 \log (t_0/s_0)$. A smooth curve, drawn to represent the corrected values, minor fluctuations being disregarded, gives results which may be called $S_1$.

The Mount Wilson spectroscopic parallaxes of main-sequence stars may then be reduced (on the average) to Schlesinger's trigonometric system, by multiplying by a factor $R$, where $\log R = 5(S_1 - S_0)$. This is given in the sixth column of Table 3. It differs so little from unity that it may be ignored, except in refined studies. For the fainter M dwarfs it is greater, but here the trigonometric parallaxes are far more accurate than spectroscopic parallaxes can be.

It will be noted that the values of $R$ do not exactly equal $t_0/s_0$. The curves defining the latter were drawn quite empirically. When combined, they indicate "stillstands" near A8 and Go in the decrease of absolute brightness with advancing spectral type, which seems improbable. The values of $R$ have been adjusted to give a more nearly uniform slope. Whether this adjustment has been overdone can hardly be determined from existing data.

The values $s_0$ and $t_0$ were used in the discussion described below. The small changes which would be required to fit the (later adopted) values of $R$ would have no sensible effect on the conclusions.

The general weighted mean of all the values of $t'$ for the whole main sequence is 0.1034—agreeing perfectly with that of $s'$.

The data for the giants are given in Table 4. Since these stars are much more distant, the percentage errors of the trigonometric parallaxes are large and the weights small, especially for class M. The irregularity in the values of $\overline{s'}$ arises mainly from differences in the proportions of supergiants and subgiants in the various groups. For $\overline{t'}$, accidental error explains the large variations, especially for the small groups, although the small values for the early G's may be significant. These fluctuations and the uneven distribution of the

supergiants make it impracticable to draw mean curves for $s_0$ and $t_0$. The weighted means for each of the four large groups into which the stars were combined have been adopted. The weighted mean of $s'$ for all the 732 stars is 0".0949, and of $t'$ also 0".0949. The zero points of the trigonometric and spectroscopic series are therefore in exact agreement. There is no evidence of difference for the various spectral groups, except possibly for the early G stars.

TABLE 4

OBSERVATIONAL DATA: GIANTS

| SP | $N$ | $[p]$ | $\bar{s}'$ | $\bar{t}'$ | $s_0$ | $t_0$ | $\frac{[p]B}{100}$ | $NE$ | $NE'$ | $NF$ |
|---|---|---|---|---|---|---|---|---|---|---|
| G0–G1 | 13 | 8.2 | 0".064 | 0".044 | 0".079 | 0".069 | 18.7 | 1.72 | 3.85 | 0.79 |
| G2–G3 | 24 | 11.4 | .064 | .057 | .079 | .069 | 38.4 | 5.40 | 16.48 | 4.52 |
| G4 | 28 | 8.4 | .099 | .104 | .079 | .069 | 50.0 | 2.58 | 15.90 | 3.98 |
| G5 | 32 | 12.1 | .092 | .074 | .079 | .069 | 54.5 | 1.42 | 9.55 | 1.83 |
| G6 | 42 | 15.7 | .101 | .109 | .103 | .107 | 29.8 | 2.58 | 7.58 | 1.64 |
| G7 | 38 | 13.8 | .103 | .099 | .103 | .107 | 27.1 | 2.98 | 10.45 | 3.47 |
| G8 | 40 | 13.5 | .097 | .117 | .103 | .107 | 29.0 | 1.28 | 6.96 | 1.45 |
| G9 | 26 | 9.3 | .093 | .082 | .103 | .107 | 18.6 | 0.29 | 4.77 | 0.66 |
| K0 | 58 | 20.0 | .102 | .099 | .103 | .107 | 41.8 | 3.92 | 10.38 | 3.40 |
| K1 | 69 | 22.0 | .115 | .122 | .103 | .107 | 50.6 | 3.66 | 17.70 | 5.05 |
| K2 | 52 | 12.8 | .103 | .116 | .096 | .098 | 49.5 | 2.08 | 12.46 | 1.58 |
| K3 | 52 | 12.6 | .100 | .102 | .096 | .098 | 49.6 | 0.82 | 13.07 | 1.64 |
| K4 | 45 | 9.9 | .090 | .100 | .096 | .098 | 43.5 | 1.00 | 10.36 | 1.54 |
| K5 | 70 | 18.5 | .090 | .080 | .096 | .098 | 65.9 | 1.23 | 14.00 | 1.06 |
| M0 | 32 | 7.1 | .102 | .106 | .085 | .085 | 39.4 | 0.73 | 8.26 | 0.70 |
| M1 | 21 | 4.0 | .068 | .104 | .085 | .085 | 26.3 | 1.16 | 4.72 | 0.58 |
| M2 | 34 | 7.2 | .076 | .051 | .085 | .085 | 42.2 | 1.69 | 6.00 | 1.30 |
| M3 | 27 | 3.8 | .093 | .102 | .085 | .085 | 34.8 | 0.14 | 7.42 | 0.14 |
| M4 | 20 | 1.4 | .098 | .112 | .085 | .085 | 26.7 | 0.10 | 3.12 | 0.29 |
| M5–M6 | 9 | 1.2 | 0.072 | 0.016 | 0.085 | 0.085 | 11.6 | 0.17 | 3.80 | 0.48 |

## 11. ANALYSIS FOR CORRELATION

We may now apply equations (23) to (26). The small groups of Tables 3 and 4 were at first treated separately. The results were often ragged, and some values of $Y_0^2$, and more rarely of $W_1^2$, came out negative, showing that the analysis was vitiated by the accidental accumulation of accidental errors. The results here presented depend on means for groups large enough to be statistically significant. The data for the subgroups were combined as described in section 6. Table 5 gives, for these groups, the quantities which appear in equations (24) to (26) and the derived values of $l$, $W_1$, and $Y_0$. Following the seven groups of main-sequence stars are means for

two larger groups, and for the whole. Similar data for the giants appear in the lower part of the table.

For the main-sequence stars as a whole, the effect on the reduced parallaxes of the real dispersion of their absolute magnitudes ($W_1^2$) is somewhat greater than that of the errors of the trigonometric determinations ($BX_0^2$). The latter is smallest (though still sensible) for the M dwarfs. For the A stars it is twice as great as the real dispersion. For the giants, the trigonometric errors are relatively much greater (though no worse in seconds of arc); and for the giant M's they swamp the real dispersion so heavily that its value can hardly be ascertained in this way, and the computed value of $Y_0^2$ comes out negative.

TABLE 5

CORRELATION ANALYSIS

| Sp | $N$ | $[p]$ | $W_1^2+BX_0^2$ | $BX_0^2$ | $W_1^2$ | $lW_1^2$ | $l^2(W_1^2+Y_0^2)$ | $l$ | $W_1$ | $Y_0$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | Main Sequence | | | | | | | | | |
| A1–A9 | 124 | 84.6 | 0.250 | 0.167 | 0.083 | 0.039 | 0.048 | 0.47 | 0.29 | 0.37 |
| F0–F4 | 187 | 141.8 | .289 | .149 | .140 | .075 | .053 | .54 | .37 | .21 |
| F5–F9 | 227 | 178.5 | .272 | .108 | .164 | .049 | .038 | .30 | .40 | .51 |
| G0–G4 | 186 | 149.9 | .219 | .098 | .121 | .039 | .027 | .32 | .35 | .38 |
| G5–K1 | 178 | 171.2 | .242 | .087 | .155 | .054 | .043 | .35 | .39 | .44 |
| K2–K6 | 138 | 168.1 | .132 | .063 | .069 | .023 | .022 | .33 | .26 | .37 |
| M0–M5 | 100 | 159.3 | 0.100 | 0.021 | 0.079 | 0.039 | 0.043 | 0.50 | 0.28 | 0.31 |
| A1–F9 | 538 | 404.9 | 0.274 | 0.135 | 0.139 | 0.056 | 0.046 | 0.40 | 0.37 | 0.38 |
| G0–M5 | 602 | 648.5 | 0.173 | 0.067 | 0.106 | 0.039 | 0.034 | 0.36 | 0.33 | 0.38 |
| All | 1140 | 1053.4 | 0.212 | 0.093 | 0.119 | 0.046 | 0.038 | 0.383 | 0.344 | 0.378 |
| | Giants | | | | | | | | | |
| G1–G5 | 97 | 40.1 | 1.140 | 0.696 | 0.444 | 0.277 | 0.277 | 0.62 | 0.67 | 0.52 |
| G6–K1 | 273 | 94.3 | 0.613 | 0.361 | .252 | .166 | .156 | 0.66 | .50 | .33 |
| K2–K5 | 219 | 53.8 | 0.927 | 0.670 | .257 | .108 | .095 | 0.42 | .51 | 0.53 |
| M0–M6 | 143 | 24.7 | 1.349 | 1.267 | 0.082 | 0.141 | 0.162 | (1.73) | (0.29) | ..... |
| G1–K1 | 370 | 134.4 | 0.771 | 0.462 | 0.309 | 0.199 | 0.192 | 0.64 | 0.57 | 0.39 |
| K2–M0 | 362 | 78.5 | 1.060 | 0.858 | 0.202 | 0.119 | 0.116 | 0.59 | 0.45 | 0.37 |
| All | 732 | 212.9 | 0.877 | 0.608 | 0.269 | 0.170 | 0.164 | 0.63 | 0.52 | 0.38 |

The resulting values of $W_1$ for the various main-sequence groups are in tolerable agreement. The general mean, 0.344, is the standard deviation of exp 0.46 $(M - M_0)$ divided by its own mean value. If $M - M_0$ has a normal distribution, the corresponding standard deviation of $M$ is $\pm 0.72$ mag.[8]

For the giants, the values of $W_1$ are greater (except the ill-determined one for Mo–M6). The general mean, $\pm 0.52$, corresponds to a standard deviation of 1.07 mag. in M. This larger result arises from the inclusion of supergiants and subgiants. Its numerical value depends upon the proportions of these which happened to be included in the present list, and has no physical significance.

The values of $Y_0$ are more ragged than those of $W_1$—as might be expected, since its determination involves taking another difference of squares, followed by division by $l$, which is none too precisely determined. Its mean value for the main sequence corresponds to a standard deviation $\pm 0.80$ mag. in M. The mean for the giants is almost identical.

This is a much greater error than has usually been attributed to spectroscopic determinations of absolute magnitude. The reason has been explained in section 3. The values previously adopted were the apparent dispersions, $lY_0$. For the mean of the main-sequence stars this is 0.145, and for the giants 0.240, corresponding to standard deviations $\pm 0.32$ mag. and $\pm 0.51$ mag. in M.

The factor $l$ averages considerably smaller for the main-sequence stars than for the giants, mainly because the real dispersion $W_1$ is smaller compared with the observational error $Y_0$.

## 12. STATISTICAL ERRORS

The statistical errors which might be expected, on a basis of random sampling from a homogeneous population, are given in Table 6 for the principal quantities derived in Table 5. The coefficient

$$a = \frac{W_1^2}{l(W_1^2 + Y^2)}$$

[8] If $y = \exp ax$, and $x$ has a normal distribution with standard deviation $\sigma$,

$$\bar{y} = \exp \tfrac{1}{2}a^2\sigma^2; \qquad \overline{(y-\bar{y})^2} = \exp 2a^2\sigma^2 - \exp a^2\sigma^2;$$
$$[\sigma(y)/\bar{y}]^2 = \exp a^2\sigma^2 - 1.$$

is the slope of the regression line (8) and $1/l$ that of the "impartial" line (10).

The anticipated mean errors were computed by equations (37) to (39) and converted into probable errors. The errors given for the larger groups are derived from the sums of $[pvv]/s_0^2$, etc., for the

TABLE 6

STATISTICAL PROBABLE ERRORS

| Sp | $W_1$ | $a$ | $1/l$ |
|---|---|---|---|
| | Main Sequence | | |
| A1–A9 | 0.29 ±0.037 | 0.80±0.12 | 2.14±0.51 |
| F0–F4 | .37 ± .027 | 1.41± .07 | 1.86± .18 |
| F5–F9 | .40 ± .022 | 1.29± .11 | 3.33± .32 |
| G0–G4 | .35 ± .023 | 1.42± .12 | 3.14± .33 |
| G5–K1 | .39 ± .022 | 1.27± .10 | 2.87± .26 |
| K2–K6 | .26 ± .020 | 1.03± .12 | 3.04± .45 |
| M0–M5 | 0.28 ±0.012 | 0.92±0.08 | 2.02±0.19 |
| A1–F9 | 0.37 ±0.015 | 1.23±0.06 | 2.47±0.16 |
| G0–M5 | 0.33 ±0.010 | 1.15±0.05 | 2.74±0.15 |
| All | 0.344±0.009 | 1.19±0.04 | 2.61±0.11 |
| | Giants | | |
| G1–G5 | 0.67 ±0.13 | 1.00±0.12 | 1.60±0.32 |
| G6–K1 | .50 ± .07 | 1.06± .07 | 1.52± .16 |
| K2–K5 | .51 ± .12 | 1.13± .13 | 2.37± .49 |
| M0–M6 | 0.29 ±0.66 | 0.87±0.13 | 0.58±0.72 |
| G1–K1 | 0.57 ±0.06 | 1.04±0.06 | 1.55±0.15 |
| K2–M6 | 0.45 ±0.13 | 1.02±0.10 | 1.70±0.40 |
| All | 0.52 ±0.06 | 1.03±0.05 | 1.59±0.15 |

separate smaller groups and do not include the effects of variation of $s_0$ and $t_0$ from group to group (which, however, are small).

For the main sequence the results for each group are statistically significant, and also for the giants, except for the M's. Although this group contains 143 stars, the parallaxes are so small that the errors of sampling deprive the deduced values of $W_1$ and $1/l$ of meaning.

The value of $a$, which is much less influenced by these errors, is significant.

The range of the results for the main-sequence groups shows that influences other than errors of sampling cause them to differ. If the seven values for $W_1$ are treated as independent determinations, with weights as in Table 5, the general mean is $0.340 \pm 0.016$ (P.E.). The mean value is substantially the same as before, but the probable error is 1.7 times greater, indicating that the real differences (measured by their squares) are twice the statistical. There is a fairly steady decrease in the dispersion of $W_1$ (or of the absolute magnitude) with "later" spectral type, except for the group A1–A9, which may not be typical of all stars in this spectral range, as many of them are ill-suited to spectroscopic determination of parallax by the methods employed for the redder stars.

For the regression constant $a$, the weighted mean is $1.18 \pm 0.06$. Sampling error contributes two-fifths of the squares of the discordances. This may account for a good part of the raggedness of the individual values; but the remainder seems pretty large to be explained by differences in the calibration-curves, which, for the same pair of lines, are usually nearly parallel for different spectral classes.

For $1/l$ the weighted mean is $2.69 \pm 0.16$. Once again, the mean value is not sensibly affected; but the probable error is much increased. The mean for the separate values of $l$ is $0.390 \pm 0.025$. The weighted mean for $Y_0$ is $0.377 \pm 0.025$. The error is greater than for $W_1$, as was to be expected. The corresponding statistical expectation was too troublesome to compute.

For the giants, the weighted means and probable errors are $W_1 = 0.51 \pm 0.04$, $a = 1.04 \pm 0.03$, $1/l = 1.63 \pm 0.20$. Two of these three errors are less than the anticipated values, doubtless because of accidentally favorable grouping of the errors.

## 13. FINAL REDUCTION FORMULAE

In discussing these results, we must remember the two types of calibration, of which type I seeks to obtain agreement between the mean spectroscopic and trigonometric parallaxes for stars grouped according to the spectroscopic absolute magnitude; while type II

seeks this agreement for groups selected by some impartial criterion, unaffected by the errors of either sort of observations.

As regards type I, the equality of the mean values of $\overline{s'}$ and $\overline{l'}$ (Table 4) and also the values of $a$ (Table 6) show that for the giants the calibration employed in *Mount Wilson Contribution* No. 511 requires, in the mean, no correction. Possible differences in the corrections for different spectral types are too small to be revealed by the trigonometric parallaxes. Since the latter have the lowest percentage accuracy for giants, and the published calibration has been largely based on other weighty data, such as proper motions and radial velocities, this agreement might fully as well be taken as evidence of the correctness of the system of Schlesinger's parallax catalogue as of that of the spectroscopic parallaxes.

For the main sequence, however, the trigonometric parallaxes constitute the strongest existing data. Here there is no evidence of a general difference of zero point, but the mean value of $a$ shows conclusively that the calibration would be improved if the differences $s' - \overline{s'}$ were increased by 19 per cent. It would be still better to apply this correction to the absolute-magnitude differences $S - S_0$ (sec. 2).

If the spectroscopic parallaxes should be thus adjusted, the agreement for groups selected by spectroscopic criteria (method I) would be exact in the mean. Should this correction be made, the mean value of $l$ would be increased from 0.383 to 0.456. The latter value represents the extent to which the calibrated differences in absolute magnitude are diminished by the method depending on the regression line.

It is doubtful whether any real improvement would be obtained by using the individual values of $a$ given in Table 6 for the different groups, and the value $a = 1.19$ may well be used for the whole main sequence. The corrections to the zero point indicated by the factor $R$ (Table 3) should, however, be made in precise work.

In order to obtain a calibration of type II, much larger corrections are necessary. The zero point may be treated as before; but the differences from the mean for the spectral subclass should be multiplied by 1.6 for the giants and by 2.6 for the main sequence. It is essential here to apply this correction to the absolute magnitudes;

to treat the parallaxes directly would result in negative parallaxes for supergiants brighter than absolute magnitude $-2$. In view of the importance of other data in the calibration for the giants, it is possible that a smaller correction factor might secure a better agreement, for impartially selected groups, between mean spectroscopic parallaxes and those from all geometrical methods (including proper motions, etc.). The still larger correction factor for the main sequence is, however, strongly determined, and should undoubtedly be applied. The evidence hardly justifies using factors varying with the spectral type, except as regards the zero point.

Following equation (10), we have, for the impartial calibration line, in the case of the reduced parallaxes,

$$t' - t_0 = \frac{s' - s_0}{l} .$$

Let $s''$ be the reduced spectroscopic parallax so modified as to agree, in the mean, with the trigonometric magnitudes for large impartially selected groups. Then

$$s'' = t_0 + \frac{s' - s_0}{l} = s' + \left(\frac{1}{l} - 1\right)(s' - s_1) ,$$

where

$$s_1 = s_0 + \frac{l(s_0 - t_0)}{(1 - l)} .$$

With $1/l = 2.6$,

$$\left.\begin{aligned} s'' &= s' + 1.6\ (s' - s_1) , \\ s_1 &= s_0 + 0.62(s_0 - t_0) . \end{aligned}\right\} \tag{41}$$

We have also

$$10s' = \exp 0.46\ (S - S_0) ,$$

where $S$ is the tabular spectroscopic absolute magnitude, and $S_0$ the standard value (Table 2). If $S''$ is the absolute magnitude recalibrated according to method II, and $S_1$ that corresponding to the reduced parallax $s_1$, we have, to the first order,

$$10s' = 1 + 0.46\ (S - S_0), \text{ etc.},$$

and equations (41) become

$$S'' = S + 1.6\,(S - S_1)\,. \tag{42}$$

For the main-sequence groups we find as follows:

| | Spectrum | | | | | | | All |
|---|---|---|---|---|---|---|---|---|
| | A1–A9 | F0–F4 | F5–F9 | G0–G4 | G5–K1 | K2–K6 | M0–M5 | |
| $s_0$ | 0.105 | 0.103 | 0.107 | 0.108 | 0.101 | 0.100 | 0.103 | 0.1037 |
| $l_0$ | 0.101 | 0.097 | 0.111 | 0.114 | 0.103 | 0.094 | 0.106 | 0.1040 |
| $s_1$ | 0.108 | 0.107 | 0.104 | 0.104 | 0.100 | 0.104 | 0.101 | 0.1035 |
| $S_1 - S_0$ | $+0^M17$ | $+0^M15$ | $+0^M09$ | $+0^M09$ | $0^M00$ | $+0^M09$ | $+0^M02$ | $+0^M08$ |

When the reduced parallaxes have already been calculated, equations (41) are more convenient, as they may be applied directly to the mean for a group. When individual values are desired, it is quicker to use equation (42) or the equivalent form

$$\log\left(\frac{s''}{s}\right) = 0.32\,(S - S_1)\,, \tag{43}$$

which gives the spectroscopic parallax directly.

For the giants we have similarly, taking $1/l = 1.60$:

| | Spectrum | | | | All |
|---|---|---|---|---|---|
| | G1–G5 | G6–K1 | K2–K5 | M0–M6 | |
| $s_0$ | 0.079 | 0.103 | 0.096 | 0.085 | 0.0946 |
| $l_0$ | 0.069 | 0.107 | 0.098 | 0.085 | 0.0950 |
| $s_1$ | 0.096 | 0.096 | 0.093 | 0.085 | 0.0939 |
| $S_1 - S_0$ | $-0^M09$ | $-0^M09$ | $-0^M15$ | $-0^M35$ | $-0^M14$ |

It is obviously sufficient to use the mean values throughout, so that

$$s'' = s' + 0.6\,(s' - 0.094)\,,$$

$$S'' = S + 0.6\,(S - S_1)\,,$$

$$\log\frac{s''}{s} = 0.12\,(S - S_1)\,.$$

### 14. NUMERICAL TESTS

As a practical test of the validity of these formulae, the stars of each spectral subclass were arranged (*a*) in order of the reduced spectroscopic parallaxes $s'$, and (*b*) in order of the similarly reduced proper motions, $\mu' = f\mu$; and weighted means of $s'$ and $t'$ were taken for four groups of substantially equal numbers. The first grouping should give a correlation of type I. If $T$ is the tangential velocity of the star (in A.U./yr), we have

$$\log (10\mu') = \log T + 0.2\,(M - S_0)\,.$$

Grouping by $\mu'$ is independent of the spectroscopic errors, and therefore of type II. Despite the large variations to be expected in $T$, it should give a fairly good separation according to M.

The results for the main sequence are given in Table 7.

These results are plotted in Figure 1, with the means for grouping by $\mu'$ as dots, and those by $s'$ as crosses, and also the "impartial" line (10) and the regression line (8)—the latter dotted—as determined by the constants for each group (Table 6).

The agreement of the means by $\overline{s'}$ with the regression lines is in all cases close, as it should be, since these are derived from the same data. For the means according to $\mu'$ the agreement is unforced. Though the results are ragged for some groups, the points invariably indicate slopes much greater than those of the regression lines, and, in general, agree with the impartial lines as well as might be expected.

A least-squares solution for the four groups in the last column, with equal weights (which suffices), gives

$$t' - \overline{t'} = 2.29\,(s' - \overline{s'})\,.$$

The residuals indicate a probable error $\pm 0.033$ for this coefficient—which is doubtless too small.

From these diagrams, as well as from the analyses previously discussed, it appears beyond all doubt that the differences of the spectroscopic absolute magnitudes (or the reduced spectroscopic parallaxes $s'$) from the normal values for the spectral subclass concerned require a small increase to obtain the best calibration by the regres-

sion method, and a very large increase, by a factor of about 2.5, to represent the deviations in groups selected impartially.

TABLE 7

GROUP MEANS: MAIN SEQUENCE

| Group | | A1–A9 | F0–F4 | F5–F9 | G0–G4 | G5–K1 | K2–K6 | M0–M5 | All |
|---|---|---|---|---|---|---|---|---|---|
| | | Grouping by $s'$ | | | | | | | |
| 1 | $N$ | 29 | 44 | 58 | 43 | 40 | 33 | 23 | 270 |
| | $\bar{s}'$ | 0.077 | 0.075 | 0.078 | 0.089 | 0.080 | 0.082 | 0.079 | 0.080 |
| | $\bar{l}'$ | 0.078 | 0.060 | 0.074 | 0.074 | 0.076 | 0.073 | 0.086 | 0.074 |
| 2 | $N$ | 32 | 48 | 57 | 48 | 47 | 35 | 27 | 294 |
| | $\bar{s}'$ | 0.099 | 0.100 | 0.100 | 0.103 | 0.096 | 0.092 | 0.093 | 0.098 |
| | $\bar{l}'$ | 0.100 | 0.090 | 0.110 | 0.104 | 0.099 | 0.087 | 0.096 | 0.098 |
| 3 | $N$ | 32 | 49 | 57 | 48 | 46 | 35 | 26 | 293 |
| | $\bar{s}'$ | 0.115 | 0.113 | 0.110 | 0.114 | 0.106 | 0.101 | 0.107 | 0.109 |
| | $\bar{l}'$ | 0.106 | 0.108 | 0.112 | 0.125 | 0.110 | 0.099 | 0.104 | 0.109 |
| 4 | $N$ | 31 | 46 | 55 | 47 | 45 | 35 | 24 | 283 |
| | $\bar{s}'$ | 0.135 | 0.135 | 0.128 | 0.131 | 0.126 | 0.118 | 0.131 | 0.128 |
| | $\bar{l}'$ | 0.119 | 0.140 | 0.138 | 0.141 | 0.138 | 0.112 | 0.136 | 0.133 |
| | | Grouping by $\mu'$ | | | | | | | |
| 1 | $N$ | 29 | 44 | 56 | 43 | 40 | 33 | 23 | 268 |
| | $\bar{s}'$ | 0.095 | 0.094 | 0.090 | 0.104 | 0.093 | 0.092 | 0.094 | 0.094 |
| | $\bar{l}'$ | 0.068 | 0.074 | 0.082 | 0.092 | 0.086 | 0.085 | 0.089 | 0.084 |
| 2 | $N$ | 32 | 48 | 57 | 47 | 46 | 34 | 26 | 290 |
| | $\bar{s}'$ | 0.105 | 0.098 | 0.103 | 0.110 | 0.102 | 0.099 | 0.100 | 0.102 |
| | $\bar{l}'$ | 0.095 | 0.087 | 0.108 | 0.115 | 0.101 | 0.091 | 0.107 | 0.101 |
| 3 | $N$ | 32 | 49 | 57 | 47 | 47 | 35 | 27 | 294 |
| | $\bar{s}'$ | 0.104 | 0.113 | 0.114 | 0.115 | 0.098 | 0.099 | 0.104 | 0.106 |
| | $\bar{l}'$ | 0.110 | 0.114 | 0.118 | 0.122 | 0.110 | 0.096 | 0.104 | 0.110 |
| 4 | $N$ | 31 | 46 | 55 | 48 | 45 | 34 | 24 | 283 |
| | $\bar{s}'$ | 0.119 | 0.118 | 0.109 | 0.116 | 0.116 | 0.104 | 0.114 | 0.113 |
| | $\bar{l}'$ | 0.137 | 0.121 | 0.131 | 0.137 | 0.135 | 0.105 | 0.126 | 0.126 |

It is not worth while to make a similar graphical treatment of the giants, since the percentage errors of the trigonometric parallaxes are

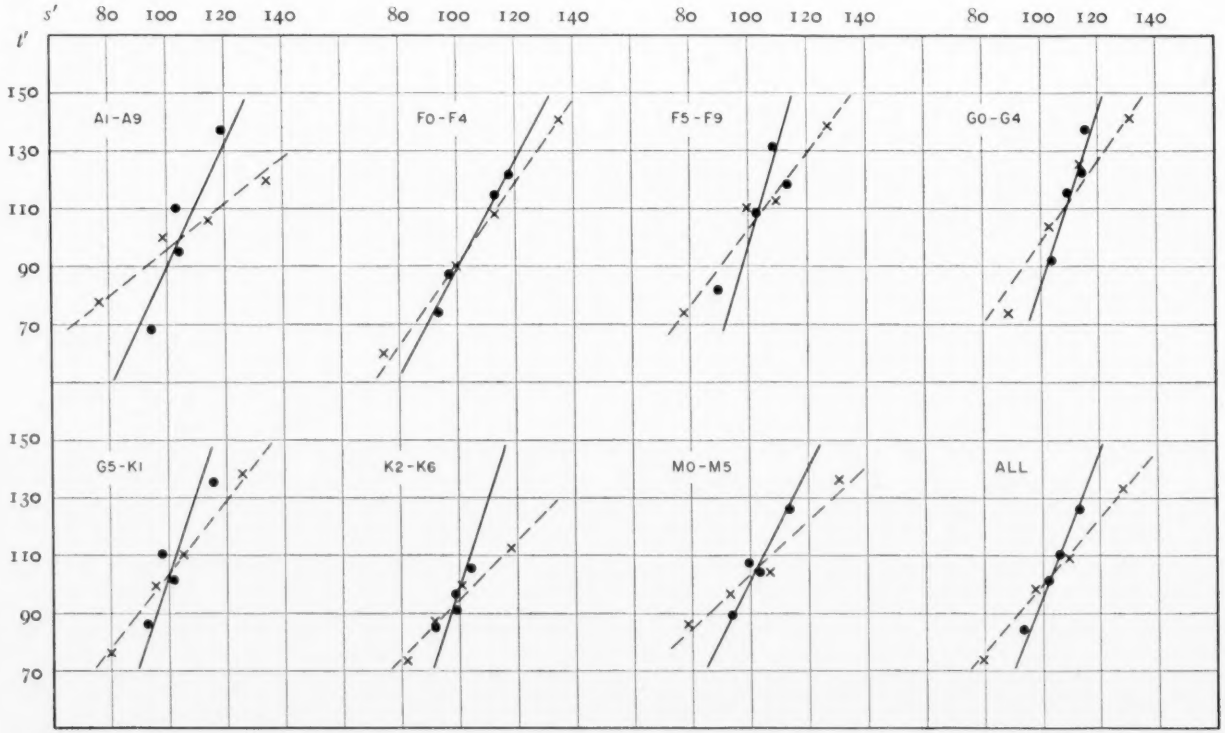


FIG. 1.—Group means, main sequence

so much larger. The results of Tables 5 and 6 show, however, that in this case also the spectroscopic method is successful in finding not only the mean absolute magnitudes for the spectral subclass but the deviations from this mean. The correcting factors are, in fact, considerably smaller than for the main sequence.

### 15. ERRORS OF TRIGONOMETRIC PARALLAXES

One result of the present investigation has been to confirm the reliability of both trigonometric and spectroscopic determinations of parallax. This is conspicuously the case for the trigonometric parallaxes of giants. Examined individually, the determinations of different observers for the same star differ by so large a percentage as to raise doubt as to their significance; but the detailed statistical discussion shows that they are substantially as good as they claim to be (that is, as is indicated by the probable errors given in *Schlesinger's Catalogue*). As a further test of this, an analysis was made of the trigonometric determinations for the hardest case, the supergiants. Trigonometric parallaxes are available for 41 stars in the Mount Wilson list of classes cGo to cK5, and for 16 others (all but two of class M) whose spectroscopic absolute magnitudes (brighter than $-1.5$) mark them as objects of similar character.

It was assumed that all these stars were of the same absolute magnitude (adopted as $-2.1$ after a preliminary discussion). If $s_0$ is the parallax corresponding to this assumption, and $t$ the observed trigonometric value, the residuals $t - s_0$ should exaggerate the trigonometric errors. For each star, this ratio $(t - s_0)/r$ was taken (where $r$ is the probable error given in *Schlesinger's Catalogue*). The individual values of this ratio range from $-3.8$ to $+3.5$. Their arithmetic mean is $-0.13$, and their mean square value, 1.46, as against $(1/0.6745)$, or 1.48, to be expected in a normal distribution. The observed parallaxes then agree with the assumed value slightly *better* than would be expected from their catalogued probable errors alone, leaving a negative value for the effects of real differences in absolute magnitude.

The statistical probable error of a determination of probable error for 57 entries is, however, $\pm 9$ per cent, so that no weight is to be attached to this result. The probable error of the mean of 57 deter-

minations is $\pm 0.13r$, so that the difference of the arithmetic mean from zero is also without significance.

It is evident, however, that the tabular probable errors of *Schlesinger's Catalogue* are, on the average, reliable measures of the accuracy of the observations.

In conclusion, it is a pleasure to acknowledge our indebtedness to Messrs. Adams and Joy for unpublished data regarding the calibration-curves (sec. 8) and the distribution of the tabular parallaxes (sec. 10); and also to Miss Myrtle Richmond for material aid in the calculations described in section 14.

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# THE PHOTOGRAPHIC DETERMINATION OF STELLAR PARALLAXES WITH THE 60- AND 100-INCH REFLECTORS*

## SIXTEENTH SERIES

ADRIAAN VAN MAANEN

ABSTRACT

*Trigonometric parallaxes determined with the 60- and 100-inch reflectors.*—Results are given for 25 fields including 28 stars, 15 of which have photographic absolute magnitudes of +10 or fainter; 3 are even +15 or fainter. One white dwarf is included, AC+70°8247, for which the absolute magnitude is +11.9. For U Geminorum and SS Cygni the absolute parallaxes are +0".010 and −0".012, respectively.

The present paper gives the results for 28 stars in 25 fields, which have recently been measured for parallax. The work has been carried out in the same way as for the previously published series.[1] The results are given in Table 1.

Several of the stars require comment. The first star in the list was found by Kuiper to be a close double, $\Delta = 0''.70$, $\Delta m = 0.3$.[2] The components have absolute magnitudes of +13.1 and +13.4, respectively.

BD+23°123 is an R-type star with, according to Sanford, a velocity of 232 km/sec and dwarf characteristics in the spectrum. Using the mean of the parallax determinations by McCormick and Mount Wilson, we find an absolute visual magnitude of +5.2.

U Geminorum and SS Cygni were put on the program as the two examples of their type of variation which have fairly large motions; from these Parenago and Kukarkin[3] derived parallaxes of 0".016 and 0".038, respectively. The absolute parallaxes derived here are +0".010 and −0".012, and it is doubtful if the stars should be considered as dwarfs.

* *Contributions from the Mount Wilson Observatory, Carnegie Institution of Washington*, No. 590.

[1] *Mt. Wilson Contr.*, Nos. 111, 1916; 136, 1917; 158, 1918; 182, 1920; 204, 1921; 237, 1922; 270, 1923; 290, 1925; 321, 1926; 356, 1928; 391, 1929; 435, 1931; 468, 1933; 506, 1935; 553, 1936.

[2] *Pub. A.S.P.*, **47**, 282, 1935.

[3] *Veränderliche Sterne, V.F.A., Gorki*, **4**, 249, 1934.

A star close to Ross 110 ($\Delta = \pm 5''$) was found to have the same large proper motion of $0''.70$. Its magnitude is about 0.4 fainter than that of Ross 110.

Near OΣ 285 ($\Delta = 50''$) a star was found which shares the motion of the principal double; apparently this is a triple system. The motions in declination were also measured; and the results are $\mu_\delta = +0''.051$ and $+0''.052$, respectively.

TABLE 1

| Object | α 1900 | δ 1900 | Phot. Mag. | Mt. Wilson Spectrum | Relative Parallax | $\mu_\alpha$ Rel. | P.E. | No. of Exp. | Instrument |
|---|---|---|---|---|---|---|---|---|---|
| Anonymous | $0^h\ 7^m42^s$ | $+68°\ 47'$ | 13.7 | ..... | $+0''.056$ | $+0''.704$ | 7 10 | 18 | 60 |
| BD+23°123 | 0 48 54 | +23 32 | 8.8† | R3 | − .005 | +0.139 | 6 1 | 18 | 60 |
| Ross 17 | 2 4 12 | +34 59 | 15.4 | ..... | + .043 | −0.314 | 6 1 | 20 | 100 |
| Σ 228 | 2 7 42 | +47 1 | 6.31 | F2 | + .012 | −0.078 | 7 2 | 18 | 60 |
| Ross 60 | 5 57 39 | +26 10 | 15.1 | ..... | + .046 | +0.339 | 5 2 | 20 | 100 |
| U Geminorum | 7 49 10 | +22 16 | Var. | ..... | + .011 | −0.042 | 7 10 | 18 | 60 |
| vM- W 48* | 9 10 24 | +14 41 | 16.24 | ..... | + .012 | +0.018 | 5 2 | 18 | 100 |
| Ross 96 | 9 49 45 | +22 58 | 14.2 | ..... | + .006 | +0.003 | 3 1 | 16 | 100 |
| Ross 99 | 10 30 8 | + 8 3 | 13.1 | ..... | + .012 | +0.254 | 8 3 | 20 | 60 |
| Ross 110 A | 11 22 53 | +57 17 | 15.5 | ..... | + .028 | +0.556 | 12 5 | 16 | 60 |
| Ross 110 B | 11 22 53 | +57 17 | 15.9 | ..... | + .035 | +0.552 | 17 8 | 16 | 60 |
| OΣ 285 A+B | 14 41 42 | +42 48 | 7.66 | F5 | + .029 | −0.055 | 10 2 | 16 | 60 |
| OΣ 285 C | 14 41 37 | +42 48 | 12.6 | ..... | + .001 | −0.050 | 6 1 | 14 | 60 |
| Ross 508 | 15 19 18 | +17 51 | 15.4 | ..... | + .087 | −0.376 | 9 3 | 20 | 100 |
| Radcl. 60, 600 | 15 57 46 | +30 27 | 14.1 | ..... | + .059 | −0.311 | 3 5 | 18 | 60 |
| OΣ 323 | 17 2 18 | +47 6 | 9.08 | K3 | + .004 | −0.015 | 8 1 | 20 | 60 |
| AC+70°8247 | 19 1 8 | +70 30 | 13.1 | ..... | + .047 | +0.064 | 4 7 | 20 | 60 |
| vM- W 24* | 19 59 18 | +29 36 | 16.5 | ..... | + .084 | +0.682 | 6 2 | 16 | 100 |
| vM- W 71* | 20 12 23 | +15 8 | 15.3 | ..... | − .003 | +0.068 | 4 1 | 16 | 100 |
| vM- W 72* | 20 12 24 | +15 8 | 18.3 | ..... | − .005 | +0.064 | 5 2 | 16 | 100 |
| Ross 188 | 20 27 42 | +38 12 | 15.0 | ..... | + .063 | +0.207 | 4 7 | 20 | 60 |
| ADS 14314A | 20 44 13 | +38 55 | 7.43 | cA7e | − .017 | +0.005 | 5 1 | 18 | 60 |
| Luyten 983 | 21 33 20 | −24 35 | 14.8 | ..... | + .066 | +1.008 | 10 3 | 20 | 100 |
| Ross 201 | 21 37 6 | +53 33 | 16.3 | ..... | + .041 | +0.369 | 8 10 | 20 | 60 |
| vM- W 112* | 21 37 10 | − 0 4 | 17.38 | ..... | + .033 | −0.213 | 12 5 | 20 | 100 |
| SS Cygni | 21 38 47 | +43 8 | Var. | Bpev | − .017 | +0.112 | 8 8 | 20 | 60 |
| Anonymous | 21 51 2 | +47 54 | 14.7 | ..... | + .018 | −0.332 | 5 2 | 20 | 60 |
| Lal. 43751 | 22 19 29 | +38 4 | 6.70 | F8‡ | +0.025 | +0.254 | 10 2 | 16 | 60 |

*From *Mt. W. Contr.*, No. 412, 1930. †Visual magnitude.
‡Spectrum from *Henry Draper Catalogue*.

Ross 508, vM- W 24, which is a distant companion of Lalande 38380 ($\Delta = 3'$), and vM- W 112 have absolute photographic magnitudes of +15.1, +15.4, and +15.0, respectively. *Mount Wilson Contribution* No. 507[4] gives a list of 12 stars whose photographic abso-

[4] *Ap. J.*, **81**, 152, 1935.

lute magnitudes are +15 or fainter. The companion of Procyon should be removed from this list, since Kuiper has recently found it to be 2.7 mag. brighter than as given there. The companion of Wolf 860 also should be omitted, as the absolute magnitude is +14.9. The stars given in the accompanying table should, however, be added to

| Object | $\alpha$ 1900 | $\delta$ 1900 | $\beta$ | Pg. $m$ | $\mu$ | $\pi$ | Pg. $M$ |
|---|---|---|---|---|---|---|---|
| Ross 92..... | $9^h35^m24^s$ | +22°29′ | +48° | 15.7 | 0.72 | +0.089 | +15.4 |
| Ross 508..... | 15 19 18 | +17 51 | +52 | 15.4 | 1.23 | + .088 | +15.1 |
| Companion of Lalande 38380... | 19 59 18 | +29 36 | − 2 | 16.5 | 0.92 | + .061 | +15.4 |
| vM- W 112..... | 21 37 10 | − 0 4 | −38 | 17.4 | 0.45 | +0.034 | +15.0 |

the list. We therefore know at present 14 stars with photographic absolute magnitudes +15.0 or fainter; three of these stars are companions. The star mentioned by Luyten in *Astronomical Journal*, No. 1070 (**46,** 133, 1937), may fall in this class too, but the preliminary parallax derived by Luyten is too uncertain to include the star at present.

OΣ 323 is apparently an optical double; the change in distance and position angle shows a relative motion in a straight line, without any indication of orbital motion.

AC+70°8247 is the white dwarf, found by Kuiper and discussed by him in *Publications of the Astronomical Society of the Pacific*, **47,** 307, 1935. The mean of the absolute parallaxes derived at Greenwich and here is +0″.057, giving an absolute photographic magnitude of +11.9.

Although the proper motions in $\alpha$ for vM- W 71 and 72 as published in *Mount Wilson Contribution* No. 412 were +0″.079 and +0″.118, respectively, it should be noted that the stars were 21′ from the center of the plates then measured, which were taken at the Newtonian focus of the 60-inch reflector. The present measures give +0″.068 and +0″.064, respectively; and the pair is probably a distant system of the 61 Cygni type.

BD+48°4235 = ADS 14314, a star of the $\alpha$ Cygni type. The 13.7 companion at $\Delta = 2''.3$ was not measurable, as it was covered by the sector.

The motion and parallax of Ross 201 are very much the same as for Ross 200; this pair, also, may therefore be a system of the 61 Cygni type:

| | | | | | |
|---|---|---|---|---|---|
| Ross 200..... | $21^h36^m33^s$ | $+53°41'$ | $\mu=0''.60$ in | $p=64°$ | $\pi=+0''.039$ |
| Ross 201..... | 21 37 6 | +53 33 | 0.55 | 75 | +0.045 |

In recent years I have tried to determine the parallaxes of those stars which presented a fair chance of having faint absolute magnitudes; and of the stars on my program thus far finished, 114 fall under this heading. Faintness has been no objection, since with the 100-inch reflector we can, with somewhat increased exposures, reach the eighteenth photographic magnitude. Altogether we have found but five negative parallaxes among the 114 stars; they include, moreover, 15 companions. Omitting both these groups, we find the absolute magnitudes to be distributed as shown in the accompanying table.

| $M$ | $n$ | $M$ | $n$ |
|---|---|---|---|
| $<+9.0$..... | 8 | $+13.0-13.9$..... | 18 |
| $+9.0-9.9$..... | 6 | $14.0-14.9$..... | 9 |
| $10.0-10.9$..... | 10 | $+15.0-15.9$..... | 8 |
| $11.0-11.9$..... | 14 | $\geqq+16.0$..... | 2 |
| $+12.0-12.9$..... | 20 | | |

The maximum frequency falls at about +12.7 photographic or, as practically all these stars are probably of type M, at +10.9 visual. If now we look at Figure 1 of *Mount Wilson Contribution* No. 562,[5] it is evident that, of the stars with large proper motions as a whole, we know only the brighter ones. The expected maximum is about 2 mag. fainter than the observed maximum. We may therefore conclude that, if we knew also the absolute magnitudes of the fainter stars of large proper motion, the maximum frequency would be shifted about 2 mag. and therefore be +12.9, a good confirmation of the result derived in *Mount Wilson Contribution* No. 562.[5]

CARNEGIE INSTITUTION OF WASHINGTON
MOUNT WILSON OBSERVATORY
January 1938

[5] *Ibid.*, **85**, 26, 1937.

# ON THE POSSIBILITY OF DETECTING MOLECULAR HYDROGEN AND NITROGEN IN PLANETARY AND STELLAR ATMOSPHERES BY THEIR ROTATION-VIBRATION SPECTRA

GERHARD HERZBERG

ABSTRACT

The detection of molecular hydrogen and nitrogen in planetary or stellar spectra, hitherto deemed impossible, can be carried out by means of the rotation-vibration spectrum of these molecules. Though $H_2$ and $N_2$, as is well known, have no ordinary rotation-vibration spectra (since their dipole moment is zero), they do have rotation-vibration spectra, owing to their *quadrupole moment.*

In the case of $H_2$ the 1—0 band of this quadrupole rotation-vibration spectrum, according to calculations of James and Coolidge, is $8.1 \times 10^{-9}$ times as intense as the 1—0 band of the ordinary rotation-vibration spectrum of *HCl.* The *minimum absorbing layer necessary to detect the 1—0, 2—0, and 3—0 bands is found to be 2.5, 2.7, and 13.0 km atm., respectively.* This is of the order of magnitude probably available in the atmospheres of the major planets. A table of the positions of the lines of the 1—0, 2—0, 3—0, and 4—0 bands as predicted from the ultraviolet $H_2$ spectrum is given. The band most favorable for detection is the 3—0 band at 8500 A. Failure to observe this band would at least give an *upper limit* for the amount of $H_2$ present in the atmospheres of the major planets or of low-temperature stars.

For $N_2$ the predicted positions of the $Q$ branches of the bands are given. Their detection will probably be more difficult than the detection of the $H_2$ bands.

A further possibility of detecting molecular hydrogen and nitrogen is by the *ordinary* rotation-vibration spectrum of the isotopic molecules $HD$ and $N^{14}N^{15}$, which are always present in natural hydrogen and nitrogen, respectively.

## I. INTRODUCTION

It has, up to the present, always been considered impossible to detect molecular hydrogen or nitrogen in planetary or stellar atmospheres.[1] The band systems of $H_2$ and $N_2$ in the visible and the near ultra-violet regions have highly excited electronic states (>6 volts above the ground state) as their lower states and consequently cannot, in general, appear in absorption. If in a high-temperature star the thermal energy would be sufficient to excite these levels, at the same time it would be sufficient to dissociate the molecules [$D(H_2) = 4.45$, $D(N_2) = 7.35$ volts], and again no molecular absorption would occur.

On the other hand, according to Wildt,[2] Russell,[1] and others, it seems necessary to assume the existence of large amounts of molec-

[1] H. N. Russell, *Nature*, **135**, 219, 1935. [2] *Ibid.*, **134**, 418, 1934.

ular hydrogen in the atmospheres of the major planets and also a certain amount of molecular nitrogen, as indicated by the presence of $CH_4$ and $NH_3$ in these atmospheres. Also, the atmospheres of the cooler stars, according to Russell,[3] contain considerable amounts of $H_2$ and $N_2$. It would consequently be of great interest if it were possible to detect $H_2$ and $N_2$ spectroscopically in planetary and stellar atmospheres.

It is the object of this paper to point out a possibility of detecting the presence of sufficiently large amounts of molecular hydrogen and nitrogen in planetary and stellar atmospheres by their rotation-vibration spectra.

## II. GENERAL DISCUSSION OF QUADRUPOLE ROTATION-VIBRATION SPECTRA

It is generally considered that a rotation-vibration spectrum of a homonuclear diatomic molecule, such as $H_2$ or $N_2$, does not exist. This consideration is based on the fact that a homonuclear diatomic molecule has no dipole moment. But, apart from an alternating dipole moment, an alternating quadrupole moment may also lead to emission or absorption of radiation (quadrupole radiation); and certainly a homonuclear molecule has a quadrupole moment which varies during the vibration of the molecule. It is true that quadrupole radiation is very much less intense than ordinary dipole radiation, and therefore rotation-vibration spectra due to quadrupole radiation have thus far never been observed and have not even been discussed in any detail. On the other hand, as is well known, quadrupole radiation is responsible for certain electron transitions in atoms like those giving rise to the nebular lines.

Ordinary infrared rotation-vibration spectra, such as those of $HCl$, can be obtained with absorbing layers ranging from a fraction of a millimeter (fundamentals) to several meters at atmospheric pressure (higher harmonics). It seems at least possible that with layers of many kilometers (reduced to atmospheric pressure), as are probably available in the major planets, quadrupole rotation-vibration spectra may be observed and, conversely, may be used to prove the presence of molecular hydrogen and nitrogen.

[3] *Ap. J.*, **79**, 317, 1934.

Therefore, it appears to be worth while to derive the positions of the absorption lines to be expected and also the absorbing layers required. From the data on the electronic band spectra of $H_2$ and $N_2$ the positions of the lines can be derived if the selection rules for quadrupole radiation are known.

Figure 1 gives schematically the rotational levels of two different vibrational levels of a homonuclear molecule like $H_2$ or $N_2$ in the ground electronic state, which for both molecules is a $^1\Sigma_g^+$ state. The even rotational levels are symmetric; the odd are antisymmetric in the nuclei. The former have the larger statistical weight in $N_2$; the latter, in $H_2$. There is always the strict selection rule that antisymmetric levels do not combine with symmetric levels, and vice versa. Therefore, a dipole transition between the two levels is not possible, since for dipole radiation we also have the selection rule $\Delta J = \pm 1$. But for quadrupole radiation[4] $\Delta J = 0, \pm 1, \pm 2$. Of these, $\Delta J = 0$ and $\Delta J = \pm 2$, as may be seen from the diagram, do not contradict the symmetry selection rule. These transitions, therefore, may occur as quadrupole radiation of homonuclear molecules. The transitions with $\Delta J = 0$ will form a $Q$ branch (but $J = 0 \rightarrow J = 0$ is forbidden), the transitions with $\Delta J = +2$ an $S$ branch, and those with $\Delta J = -2$ an $O$ branch.

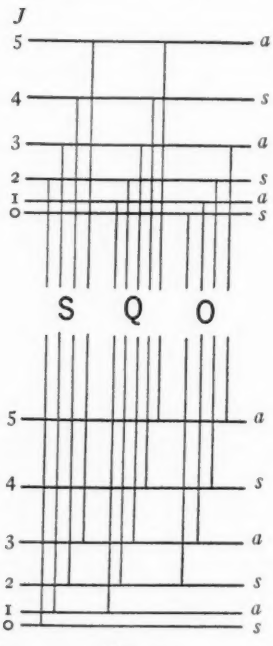


FIG. 1

Just as the intensity of dipole radiation for heteronuclear molecules like *HCl* depends upon the rate of change of the dipole moment during the vibration, the intensity of the quadrupole radiation of homonuclear molecules depends on the rate of change of the quadrupole moment.

[4] Cf., e.g., A. Rubinowicz and J. Blaton, *Ergebn. Ex. Naturwiss.* **11**, 176, 1932.

### III. THE QUADRUPOLE ROTATION-VIBRATION SPECTRUM OF $H_2$

The most complete data on the rotational and vibrational levels of the ground state of $H_2$ are given by Jeppesen.[5] From his data the wave numbers and wave lengths of the lines of the 1 — 0, 2 — 0 3 — 0, and 4 — 0 bands can be accurately calculated, taking account of the foregoing selection rules. The predicted values are given in Table 1.

TABLE 1

PREDICTED QUADRUPOLE ROTATION-VIBRATION BANDS OF HYDROGEN

| | 1—0 Band | | | | | | 2—0 Band | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $J$ | $S$ Branch | | $Q$ Branch | | $O$ Branch | | $J$ | $S$ Branch | | $Q$ Branch | | $O$ Branch | |
| | $\nu$(cm$^{-1}$) | $\lambda$(A) | $\nu$(cm$^{-1}$) | $\lambda$(A) | $\nu$(cm$^{-1}$) | $\lambda$(A) | | $\nu$(cm$^{-1}$) | $\lambda$(A) | $\nu$(cm$^{-1}$) | $\lambda$(A) | $\nu$(cm$^{-1}$) | $\lambda$(A) |
| 0 | 4498 | 22228 | ..... | ..... | ..... | ..... | 0 | 8403 | 11898 | ..... | ..... | ..... | ..... |
| 1 | 4713 | 21213 | 4155 | 24066 | ..... | ..... | 1 | 8101 | 11624 | 8072 | 12386 | ..... | ..... |
| 2 | 4917 | 20333 | 4143 | 24129 | 3807 | 26263 | 2 | 8782 | 11384 | 8048 | 12422 | 7729 | 12935 |
| 3 | 5108 | 19571 | 4126 | 24229 | 3568 | 28019 | 3 | 8945 | 11176 | 8014 | 12475 | 7485 | 13357 |
| 4 | 5286 | 18914 | 4103 | 24369 | 3339 | 29941 | 4 | 9090 | 10999 | 7968 | 12547 | 7234 | 13820 |
| 5 | 5448 | 18351 | 4074 | 24541 | 3091 | 32342 | 5 | 9213 | 10851 | 7911 | 12637 | 6979 | 14325 |

| | 3—0 Band | | | | | | 4—0 Band | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $J$ | $S$ Branch | | $Q$ Branch | | $O$ Branch | | $J$ | $S$ Branch | | $Q$ Branch | | $O$ Branch | |
| | $\nu$(cm$^{-1}$) | $\lambda$(A) | $\nu$(cm$^{-1}$) | $\lambda$(A) | $\nu$(cm$^{-1}$) | $\lambda$(A) | | $\nu$(cm$^{-1}$) | $\lambda$(A) | $\nu$(cm$^{-1}$) | $\lambda$(A) | $\nu$(cm$^{-1}$) | $\lambda$(A) |
| 0 | 12080 | 8276.2 | ..... | ..... | ..... | ..... | 0 | 15530 | 6437.2 | ..... | ..... | ..... | ..... |
| 1 | 12261 | 8153.9 | 11760 | 8501.3 | ..... | ..... | 1 | 15695 | 6369.8 | 15222 | 6567.7 | ..... | ..... |
| 2 | 12420 | 8049.4 | 11725 | 8526.3 | 11423 | 8752.2 | 2 | 15832 | 6314.8 | 15176 | 6587.5 | 14890 | 6713.9 |
| 3 | 12556 | 7962.4 | 11674 | 8563.9 | 11173 | 8947.9 | 3 | 15940 | 6272.0 | 15108 | 6617.3 | 14635 | 6831.1 |
| 4 | 12667 | 7892.5 | 11605 | 8614.2 | 10911 | 9162.7 | 4 | 16018 | 6241.3 | 15017 | 6657.2 | 14362 | 6961.1 |
| 5 | 12753 | 7839.2 | 11521 | 8677.5 | 10639 | 9396.8 | 5 | 16066 | 6222.5 | 14905 | 6707.3 | 14073 | 7103.9 |

Just as in the case of ordinary rotation-vibration spectra, it is to be expected that the intensity of the bands will decrease in the series 1 — 0, 2 — 0, 3 — 0, 4 — 0. As to the absolute intensity of the bands, in the case of $H_2$ a fairly reliable estimate can be obtained, as is shown in a paper by James and Coolidge,[6] who very kindly undertook the calculation of the rate of change of the quadrupole moment, using their previous calculations on the ground state of $H_2$. From their results it follows that the intensity of the 1 — 0 $H_2$ band

[5] *Phys. Rev.* **44**, 165, 1933.

[6] *Ap. J.*, **87**, 438, 1938.

is about $8.1 \times 10^{-9}$ times that of the $1 - 0$ *HCl* band. The $2 - 0$ and $3 - 0$ bands of $H_2$ are found by James and Coolidge to have 0.91 and 0.19 times the intensity of the $1 - 0$ band. Since the *HCl* $1 - 0$ band can be obtained with an absorbing length of 0.01 cm at atmospheric pressure,[7] one would conclude that the $1 - 0$, $2 - 0$, and $3 - 0$ $H_2$ bands may be obtained with absorbing layers of 12, 14, and 65 km atm., respectively.

But there is one factor that will reduce the necessary layer: in $H_2$, even at room temperature, only the first three rotational levels occur (at room temperature 90 per cent of all molecules), whereas for *HCl* about twelve levels are of importance. Thus, for $H_2$ the intensity is concentrated in fewer lines, and, therefore, the lines may be more easily detected than one would expect according to the foregoing. This effect is still more pronounced for the planetary atmospheres, since the temperature is smaller than $-100°$ C.[1] Taking $T = -123°$ C., it is found that the foregoing will give a factor[8] of 0.2, so that 2.5, 2.7, and 13.0 km atm., respectively, would be sufficient to produce the bands in absorption. It is difficult to estimate the accuracy of these figures, partly because of the uncertainty of the minimum layer required for the *HCl* bands and partly because the minimum layer for $H_2$ will depend greatly on the spectroscope and the dispersion used. The influence of the different line widths for $H_2$ and *HCl* has also to be taken into account. One would expect the line width for $H_2$ to be due to Doppler effect only (the half-width at 150° K would be 0.025 A), i.e., to be considerably smaller, than for the $1 - 0$ *HCl* band, for which there is strong collision broadening. Considering all this, it seems safe, however, to assume the possibility of an error of not more than one power of 10 on either side of the preceding estimate.

In Figure 2 the theoretical intensity distribution in an $H_2$ rotation-vibration band is represented, using the matrix elements given by James and Coolidge and assuming $T = 150°$ K. The figure refers to the $2 - 0$ band. But, of course, the relative intensity distribution for the other bands is the same.

[7] Cf. J. D. Hardy, E. F. Barker, and D. M. Dennison, *Phys. Rev.*, **42**, 279, 1932.

[8] This takes into consideration the fact that in *HCl* every line has two components, due to $HCl^{35}$ and $HCl^{37}$.

As is seen from Figure 2, in the case of planetary spectra, the lines $S(0)$, $S(1)$, and $Q(1)$ are the most likely to be observed; but with sufficient absorbing layers, the lines with $J = 2, 3$, and 4 may also be found. The observation of many of these lines in planetary spectra is, however, made impossible by other, much stronger absorptions of the atmosphere of the earth or of the planet.

Of the 1 — 0 band, only the line $S(1)$ might be sufficiently free from overlapping by the $H_2O$ telluric band at 1.9 $\mu$ and by strong methane bands extending from 2.15 to 2.45 $\mu$.[9] But with absorbing

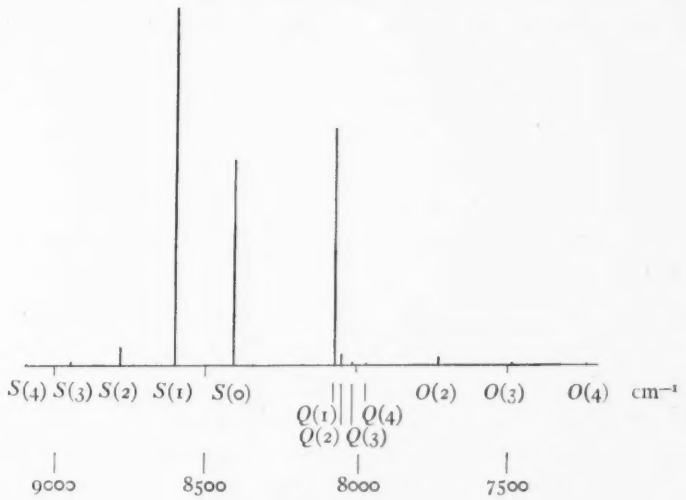


FIG. 2

layers of $CH_4$ like those on the major planets, it is likely that even this line is overlapped by methane bands (for example, by difference bands or weak parts of the main bands). Besides, an investigation in this region with sufficient dispersion would be extremely difficult.

The 2 — 0 band lies in the photographic infrared region, and the chances of its being observed are consequently better. The strong lines $S(0)$ and $Q(1)$ of this band would lie just between the two strong $H_2O$ bands at 1.1 and 1.4 $\mu$, and could therefore be observed in the spectra of low-temperature stars. But in the spectra of the major planets they would be covered by strong methane bands ex-

[9] See J. G. Moorhead, *Phys. Rev.*, **39**, 83, 1932.

tending from 1.13 $\mu$ to longer wave lengths and appearing with a few meters absorbing length only.[10] The lines $S(3)$ and $S(4)$ are free from $H_2O$ overlapping and may possibly not be obscured by $CH_4$ bands. But, as seen from Figure 2, they would be much less intense than $S(1)$ and $Q(1)$ at 150° K.

As may be seen from a spectrogram of Jupiter reproduced by Slipher,[11] there is practically no overlapping for the most intense lines $S(0)$, $S(1)$, and $Q(1)$ of the $3 - 0$ $H_2$ band. Also, it lies in a much more convenient spectral region (8000–8600 A). Therefore, except for the fact that this band is about five times less sensitive than the $2 - 0$ band, it will be particularly suitable for detecting molecular hydrogen in stellar and planetary atmospheres. Since this band consists of only a few very widely spaced single lines (with distances of more than 100 A), it will not be easy to observe it, especially because the line width is extremely small. Unless the amount of $H_2$ is much larger than the minimum amount required for observation (13.0 km atm.), considerably higher dispersion than has been heretofore used might be necessary to detect it. Failure to observe these lines, even with sufficient dispersion, would at least give an upper limit for the amount of molecular hydrogen in the atmospheres of the major planets or low-temperature stars.

The $4 - 0$ and $5 - 0$ bands would also be in a favorable position to be observed, but they would be much less intense.

## IV. THE QUADRUPOLE ROTATION-VIBRATION SPECTRUM OF $N_2$

Taking the data for the rotational constants of $N_2$ in the ground electronic state from Watson and Koontz,[12] and for the vibrational constants from Birge and Hopfield,[13] the wave numbers of the lines of the $N_2$ rotation-vibration bands have been derived. Table 2 gives the data for the zero lines of the bands only, which form the heads of the $Q$ branches. In consequence of the smaller $B$ and $a$ values for $N_2$ than for $H_2$, the lines of the $Q$ branches lie fairly close together, so that, except for very large dispersion, they will form an unresolved

[10] W. V. Norris and H. J. Unger, *ibid.*, **43**, 467, 1933, and unpublished results of the author.

[11] *M.N.*, **93**, 657, 1933.

[12] *Phys. Rev.*, **46**, 32, 1934.

[13] *Ap. J.*, **68**, 257, 1928.

broad line or band. This coalescence of most of the lines of the $Q$ branch will increase to some extent the sensitivity of the test for $N_2$. The $S$ and $O$ branches lie to either side of the $Q$ branch, with a spacing of roughly 8 $cm^{-1}$.

A reliable estimate for the intensity of the $N_2$ quadrupole rotation-vibration spectrum is thus far not possible. One may, however, expect that the intensity will be at least of the same order of magnitude as for $H_2$ but that it will drop more rapidly in the series of harmonics than for $H_2$, since the anharmonicity is smaller.

TABLE 2

PREDICTED ZERO LINES OF QUADRUPOLE ROTATION-VIBRATION BANDS OF NITROGEN

| Band | $\nu(cm^{-1})$ | $\lambda(A)$ |
|---|---|---|
| 1—0 | 2330.7 | 42894 |
| 2—0 | 4632.5 | 21581 |
| 3—0 | 6905.5 | 14477 |
| 4—0 | 9149.5 | 10927 |
| 5—0 | 11364.7 | 8796.8 |

If the intensity of absorption is large enough, the $N_2$ rotation-vibration bands are, of course, to be expected in the solar spectrum, owing to the $N_2$ content of the earth's atmosphere, which corresponds to about 5 km atm. Only if the $N_2$ content of the planetary or stellar atmospheres is comparable to, or larger than, that of the earth's atmosphere will it be possible to detect $N_2$ in them.

The frequencies of the 1 — 0 and 3 — 0 bands of $N_2$ coincide very closely with the intense fundamental $\nu_3$ of $CO_2$ and its second overtone; the 3 — 0 band also coincides with the intense $H_2O$ band at 1.4 $\mu$. Even with very high resolution, attainable only for the solar spectrum, it would be very difficult to detect the extremely weak $N_2$ lines so intermixed with the intense $CO_2$ or $H_2O$ lines.

Fortunately, the 2 — 0 and 4 — 0 $CO_2$ bands are strictly forbidden for dipole radiation, and therefore the 2 — 0 and 4 — 0 $N_2$ bands will not be disturbed by $CO_2$ overlapping. They will also be fairly free from $H_2O$ overlapping. In fact, it seems possible that a maxi-

mum at 2.16 $\mu$ in Langley and Abbot's[14] curve of the infrared solar spectrum represents the $Q$ branch of the $2 - 0$ $N_2$ band. If this identification should prove to be true, it would represent the first observed example of a quadrupole rotation-vibration spectrum; and at the same time it would show the feasibility of the detection of $N_2$ and $H_2$ in planetary and stellar atmospheres by their quadrupole rotation-vibration spectra.

The $4 - 0$ band of $N_2$ lies in the photographic infrared and would therefore be much more easily accessible, especially in planetary and stellar spectra. But, in order to be observed, it will probably require much thicker absorbing layers than the corresponding $H_2$ band, in consequence of the effect of the smaller anharmonicity mentioned above. Also, it is not certain whether in planetary spectra it will not be covered by $CH_4$ absorption, even though in the laboratory no $CH_4$ absorption has been observed in the region of the $S$ and $Q$ branches of the $N_2$ $4 - 0$ band with absorbing layers of 16 meters.

## V. DETECTION BY ORDINARY ROTATION-VIBRATION SPECTRA OF THE ISOTOPIC MOLECULES

In addition to detecting $H_2$ and $N_2$ by their quadrupole rotation-vibration spectrum, the following possibility exists: For the isotopic molecules, $HD$ and $N^{14}N^{15}$, respectively, the property "symmetric or antisymmetric in the nuclei" no longer exists. Consequently, an ordinary rotation-vibration spectrum (consisting of $P$ and $R$ branches only) can occur, which, of course, will be very weak because of the smallness of the change of dipole moment involved in the oscillations. Yet, the intensity of this weak dipole absorption would still be considerably larger than that of the corresponding quadrupole absorption. Therefore, it does not seem impossible that the small concentration of $HD$ and $N^{14}N^{15}$ molecules in natural hydrogen or nitrogen could be compensated.

Wick[15] has calculated the transition moment for the $1 - 0$ band of $HD$. From his figure and the corresponding figure for $HCl$ taken from J. L. Dunham's work,[16] an intensity ratio of $1 : 6.6 \cdot 10^5$ for the

[14] *Ann. Astrophys. Obs. Smiths. Institution*, **1**, 1900.

[15] *Accad. Lincei Atti.*, **21**, 708, 1935.

[16] *Phys. Rev.* **34**, 438, 1929.

1 — 0 bands of *HD* and *HCl* is obtained. Taking the abundance ratio of *HD* in $H_2$ equal to 1:3000, the intensity ratio of the *HD* band in natural hydrogen to the *HCl* band follows as $0.5 \cdot 10^{-9}$, and is thus of the same order as, though somewhat smaller than, the figure for quadrupole radiation. The 3 — 0 and 4 — 0 bands of *HD* would lie at 9647 A and 7421 A, respectively.

For $N^{14}N^{15}$ the abundance ratio is much more favorable, but the mass ratio is much less favorable. The bands would lie very close to the $N^{14}N^{14}$ bands given in Table 2.

In conclusion I wish to express my sincere thanks to Dr. H. N. Russell for his keen and stimulating interest in this work.

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# QUADRUPOLE ROTATION-VIBRATION SPECTRUM OF $H_2$

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ABSTRACT

The theory of the quadrupole radiation of homonuclear diatomic molecules is developed for the case of transitions between $^1\Sigma$ states. The results are applied in a theoretical treatment of the rotation-vibration spectrum of $H_2$, which, as Herzberg has suggested, may be observable in the absorption spectra of planetary atmospheres. The quadrupole moment of the system is computed as a function of the nuclear separation by the use of electronic wave functions obtained by a variational process. Matrix elements of this quantity are calculated, using the vibrational functions of the Morse and Pöschl-Teller potential-curves. Absolute values are found for the Einstein absorption coefficient B for fundamental, first, and second overtone bands, and are compared with the corresponding quantities for the dipole spectrum of *HCl*. It is found that the first and second overtone bands have intensities in the ratios 0.91 and 0.19, respectively, to that of the fundamental, and that, while this is an exceptional case, one should, in general, expect the ratios to be higher in quadrupole than in dipole bands.

## INTRODUCTION

It has been suggested by Herzberg[1] that it may be possible to detect the quadrupole rotation-vibration spectrum of $H_2$ in the atmosphere of planets or cool stars, where the great depth of the absorbing material can compensate for the small absorption coefficient. For the detailed discussion of this question, as found in Herzberg's paper, it is necessary to have a reasonably accurate quantitative idea of the magnitude of the absorption coefficients, both for the fundamental band and for the lower overtone bands. Heretofore only crude estimates of the intensity of the fundamental band have been available,[2] the possible error being of the order of a factor of 10 either way. No attempt has been made to discuss the relative strength of the overtones.

The material of the present paper falls into two main parts. In the first section we apply the general theory of quadrupole radiation to transitions between $^1\Sigma$ states, obtaining transition probabilities in terms of the rotational quantum numbers and integrals over the

[1] Preceding paper.

[2] E. Teller, *Hand- und Jahrbuch der chemischen Physik*, **9**, Part II, 52.

electronic and vibrational wave functions of the molecules. In the succeeding sections these integrations are carried out for the special case of transitions between the vibration-rotation levels of the ground state of $H_2$, and absolute values are obtained for the absorption intensity in the fundamental band and the first two overtones.

## 1. GENERAL THEORY OF QUADRUPOLE TRANSITIONS BETWEEN $^1\Sigma$ STATES

It is well known[3] that, corresponding to the expression

$$A(a, b) = \frac{64\pi^4\nu^3}{3c^3h} |(a/P/b)|^2$$

for the Einstein coefficient for spontaneous dipole radiative transitions of a system from state $a$ to state $b$, one has for spontaneous quadrupole radiative transitions

$$A(a, b) = \frac{32\pi^6\nu^5}{5c^5h} |(a/\mathfrak{N}/b)_x|^2 . \tag{1}$$

Here the Einstein coefficient $A(a, b)$ is defined through the relation

$$I_s(a, b) = h\nu \cdot N(a) \cdot A(a, b) , \tag{2}$$

$I_s(a, b)$ being the intensity of radiation arising from spontaneous transitions from $a$ to $b$, and $N(a)$ the number of systems in the initial state $a$. $\mathfrak{N}$ is a dyadic, the instantaneous quadrupole moment of the system

$$\mathfrak{N} = \sum_i e_i \vec{r}_i \vec{r}_i , \tag{3}$$

the summation being taken over all charges; and $(a/\mathfrak{N}/b)_x \equiv (a/\mathfrak{N}_x/b)$ is the matrix element between $a$ and $b$ of $\mathfrak{N}_x$, that compo-

[3] See, e.g., Condon and Shortley, *Theory of Atomic Spectra* (Cambridge, 1935), pp. 93 ff. We follow their notation in the main.

nent of $\mathfrak{R}$ which is orthogonal to the unit dyadic $\mathfrak{I}$. If we indicate the scalar of the dyadic by $\{\ \}_s$, then

$$(a/\mathfrak{R}/b)_x = \int d\tau \psi_a^* (\mathfrak{R} - \tfrac{1}{3}\{\mathfrak{R}\}_s \mathfrak{I}) \psi_b \,. \tag{4}$$

The square of this dyadic indicates the double-dot product of this quantity with its complex conjugate; in the usual notation

$$|(a/\mathfrak{R}/b)_x|^2 = \{(a/\mathfrak{R}/b)_x^* \cdot (a/\mathfrak{R}/b)_x\}_s \,. \tag{5}$$

In general, in the absence of external fields, the energy-levels of the system will have a degeneracy of $2J + 1$, associated with the different possible orientations of the total angular momentum, and the observed spectral line will arise from the transitions from the $2J_A + 1$ states of the upper level to the $2J_B + 1$ states of the lower level. For the case of natural excitation of the upper level, with the same number of systems, $N(a_i)$, in each of the degenerate states, one can define

$$I_s(A, B) = h\nu \cdot N(a_i) \sum_{i,j} A(a_i, b_j) \,, \tag{6}$$

where $A$ and $B$ represent the totality of degenerate states $a_i$ and $b_i$, respectively. Defining the Einstein coefficient for transitions between the levels $A$ and $B$ by

$$I_s(A, B) = h\nu \cdot N(A) \cdot A(A, B) \,, \tag{7}$$

one obtains

$$A(A, B) = \frac{1}{2J_A + 1} \sum_{i,j} A(a_i, b_j) \,. \tag{8}$$

Finally, defining $S(A, B)$, the strength of the line, by

$$S(A, B) = \sum_{i,j} |(a_i/\mathfrak{R}/b_j)_x|^2 \,, \tag{9}$$

one has

$$A(A, B) = \frac{32\pi^6\nu^5}{5c^5h} \frac{S(A, B)}{2J_A + 1} . \tag{10}$$

For absorption, the coefficient $B(B, A)$ is defined by

$$I(B, A) = h\nu \cdot N(B) \cdot B(B, A) \cdot \rho(\sigma) , \tag{11}$$

$I(B, A)$ being the rate of absorption of energy, and $\rho(\sigma)$ the density of isotropic, unpolarized radiation per unit difference of wave number. Between $A(A, B)$ and $B(A, B)$ there exists, of course, the same relation as for dipole transitions, and thus

$$B(B, A) = \frac{4\pi^5\nu^2}{5c^3h^2} \frac{S(A, B)}{2J_B + 1} . \tag{12}$$

The problem to be considered is the evaluation of $S(A, B)$ for transitions between $^1\Sigma$ states of a molecule. For these states the coupling of the spin to the other co-ordinates of the system is negligible; and the quantum number for total angular momentum, $J$, is identical with $K$, the quantum number for rotation of the nuclei. Separating variables in the usual way, we may then write the wave function for the molecule as

$$\psi_a = \psi_{e,v,J,M} = F_e(r; r_1, r_2, \ldots .)r^{-1}R_{evJ}(r)P_{JM}(\theta, \varphi) . \tag{13}$$

Here $F_e$ is the electronic wave function, a function of $r_1, r_2, \ldots .$, the co-ordinates of the electrons in a system of axes fixed in the molecule, which involves also as a parameter the separation of the nuclei, $r$. $r^{-1}R_{evJ}(r)$ is the vibrational function, depending on the vibrational quantum number $v$, on $J$, and on the electronic quantum numbers $e$, which determine the effective potential curve in the absence of rotation. $P_{JM}$ describes the rotation of the molecule with respect to axes fixed in space; in the present case of $\Sigma$ states it is

simply the normalized surface harmonic, which also describes the rotational motion of a single particle in a central field. We take

$$\left.\begin{aligned} P_{J,|M|} &= (-1)^M \sqrt{\frac{2J+1}{4\pi}\frac{(J-|M|)!}{(J+|M|)!}}\, P_J^{|M|}(\cos\theta) e^{i|M|\varphi}\,; \\ P_{J,-|M|} &= +\sqrt{\frac{2J+1}{4\pi}\frac{(J-|M|)!}{(J+|M|)!}}\, P_J^{|M|}(\cos\theta) e^{-i|M|\varphi}\,. \end{aligned}\right\} \tag{14}$$

Then

$$(a/\mathfrak{N}/b)_z = \int d\Omega P^*_{JM}(\theta,\varphi) P_{J'M'}(\theta,\varphi) \int_0^\infty R_{evJ}(r) R_{e'v'J'}(r)\, dr \times$$

$$\int F_e^*(r; r_1, r_2, \ldots)\, \mathfrak{N}_z F_{e'}(r; r_1, r_2, \ldots)\, \vec{dr}_1\, \vec{dr}_2 \ldots \tag{15}$$

For simplicity we shall deal with the case of $H_2$; the generalization to any other diatomic molecule is immediate and obvious. Then, in terms of $\vec{I}, \vec{J}, \vec{K}$, a set of unit vectors fixed in the molecule with $\vec{K}$ along the internuclear axis, we have, with respect to the center of gravity,

$$\begin{aligned} \mathfrak{N}_z = \sum_i \tfrac{1}{3} e_i [ & (2X_i^2 - Y_i^2 - Z_i^2)\vec{I}\vec{I} + (2Y_i^2 - X_i^2 - Z_i^2)\vec{J}\vec{J} \\ & + (2Z_i^2 - X_i^2 - Y_i^2)\vec{K}\vec{K} + 3X_iY_i(\vec{I}\vec{J} + \vec{J}\vec{I}) \\ & + 3Y_iZ_i(\vec{J}\vec{K} + \vec{K}\vec{J}) + 3X_iZ_i(\vec{I}\vec{K} + \vec{K}\vec{I})]\,. \end{aligned} \tag{16}$$

For either nucleus we have, independent of the electronic co-ordinates, $X_i^2 = Y_i^2 = 0$: $Z_i^2 = r^2/4$. Integrating over the electronic co-ordinates, and denoting the resulting averages by a bar, we have, for either electron, $\overline{X_i^2} = \overline{Y_i^2}$ and $\overline{X_iY_i} = \overline{Y_iZ_i} = \overline{X_iZ_i} = 0$, because of the axial symmetry of the wave functions. The integral over electronic co-ordinates which appears in equation (15) is thus

$$\begin{aligned} \overline{\mathfrak{N}_z} &= [2e\{\overline{(r/2)^2} + \overline{X_i^2} - \overline{Z_i^2}\}]\,[\tfrac{1}{3}\{-\vec{I}\vec{I} - \vec{J}\vec{J} + 2\vec{K}\vec{K}\}] \\ &= N(r)\mathfrak{M}(\theta,\varphi)\,. \end{aligned} \tag{17}$$

The first factor, $N(r)$, upon averaging over the vibrational functions, gives $\overline{N} = \int R_{evJ}(r)N(r)R_{e'v'J'}(r)dr$. Postponing the discussion of this quantity, we consider the averaging of the dyadic $\mathfrak{M}(\theta, \varphi)$ over the rotational functions. We transform to a stationary frame of reference, with fixed unit vectors $\vec{i}, \vec{j}, \vec{k}$, where

$$\left.\begin{aligned} \vec{I} &= -\sin\varphi\vec{i} + \cos\varphi\vec{j}\,, \\ \vec{J} &= -\cos\theta\cos\varphi\vec{i} - \cos\theta\sin\varphi\vec{j} + \sin\theta\vec{k}\,, \\ \vec{K} &= \sin\theta\cos\varphi\vec{i} + \sin\theta\sin\varphi\vec{j} + \cos\theta\vec{k}\,. \end{aligned}\right\} \tag{18}$$

We can conveniently express the result in terms of the dyadics

$$\left.\begin{aligned} \mathfrak{K}(0) &= \sqrt{\tfrac{2}{3}}\{-\tfrac{1}{2}\vec{i}\vec{i} - \tfrac{1}{2}\vec{j}\vec{j} + \vec{k}\vec{k}\}\,, \\ \mathfrak{K}(\pm 1) &= \tfrac{1}{2}\{\vec{k}\vec{i} + \vec{i}\vec{k} \pm i(\vec{k}\vec{j} + \vec{j}\vec{k})\}\,, \\ \mathfrak{K}(\pm 2) &= \tfrac{1}{2}\{\vec{i}\vec{i} - \vec{j}\vec{j} \pm i(\vec{i}\vec{j} + \vec{j}\vec{i})\}\,, \end{aligned}\right\} \tag{19}$$

which are normalized and orthogonal to each other (and to $\mathfrak{I}$). Then we easily find

$$\begin{aligned} \mathfrak{M}(\theta, \varphi) &= \sqrt{\tfrac{2}{3}}P_2^0(\cos\theta)\mathfrak{K}(0) + \tfrac{1}{3}P_2^1(\cos\theta)e^{-i\varphi}\mathfrak{K}(+1) \\ &\quad + \tfrac{1}{3}P_2^1(\cos\theta)e^{i\varphi}\mathfrak{K}(-1) + \tfrac{1}{6}P_2^2(\cos\theta)e^{-2i\varphi}\mathfrak{K}(+2) \\ &\quad + \tfrac{1}{6}P_2^2(\cos\theta)e^{2i\varphi}\mathfrak{K}(-2) \\ &= \sqrt{\frac{8\pi}{15}}\sum_{\tau=-2}^{\tau=+2}[1 - 2\delta(\tau, 1)]P_{2,\tau}(\theta, \varphi)\mathfrak{K}(-\tau)\,. \end{aligned} \tag{20}$$

Hence

$$\begin{aligned} (a/\mathfrak{N}/b)_z &= \overline{N}\sqrt{\frac{8\pi}{15}}\sum_{\tau=-2}^{\tau=+2}[1 - 2\delta(\tau, 1)]\mathfrak{K}(-\tau)\int P^*_{J,M}P_{2,\tau}P_{J'M'}d\Omega \\ &= \overline{N}\sqrt{\frac{8\pi}{15}}\sum_{\tau=-2}^{\tau=+2}[1 - 2\delta(\tau, 1)]\mathfrak{K}(\tau)\int P^*_{J',M'}P_{2,\tau}P_{J,M}d\Omega\,. \end{aligned} \tag{21}$$

To evaluate $\int P^*_{J',M'} P_{2,\tau} P_{J,M}\, d\Omega$ we express $P_{2,\tau} P_{J,M}$ as a sum of normalized surface harmonics. One easily finds the necessary formulae by iteration of

$$\cos\theta P_{J,M} = \sqrt{\frac{(J-M+1)(J+M+1)}{(2J+1)(2J+3)}} P_{J+1,M} + \sqrt{\frac{(J-M)(J+M)}{(2J-1)(2J+1)}} P_{J-1,M} ;$$

$$\sin\theta e^{i\varphi} P_{J,M} = -\sqrt{\frac{(J+M+1)(J+M+2)}{(2J+1)(2J+3)}} P_{J+1,M+1} + \sqrt{\frac{(J-M)(J-M-1)}{(2J-1)(2J+1)}} P_{J-1,M+1} ; \quad (22)$$

$$\sin\theta e^{-i\varphi} P_{J,M} = \sqrt{\frac{(J-M+1)(J-M+2)}{(2J+1)(2J+3)}} P_{J+1,M-1} - \sqrt{\frac{(J+M)(J+M-1)}{(2J-1)(2J+1)}} P_{J-1,M-1} .$$

Using the normalization and orthogonality properties of the $P_{J,M}$, one then gets immediately relations such as

$$(P_{J,M}/P_{2,0}/P_{J',M'}) =$$

$$\sqrt{\frac{5}{16\pi}} \left\{ 3\sqrt{\frac{(J-M+1)(J+M+1)(J-M+2)(J+M+2)}{(2J+1)(2J+3)^2(2J+5)}} \times \delta(J', J+2)\, \delta(M', M) + 2\frac{J(J+1)-3M^2}{(2J-1)(2J+3)} \delta(J', J)\, \delta(M', M) + 3\sqrt{\frac{(J-M)(J+M)(J-M-1)(J+M-1)}{(2J+1)(2J-1)^2(2J-3)}} \times \delta(J', J-2)\, \delta(M', M) \right\} . \quad (23)$$

From these formulae one obtains the desired values for $(a/\mathfrak{N}/b)_x$, as follows:

$\Delta J = J' - J = 2$, $\Delta M = 0$:

$$\sqrt{\frac{3}{2}}\sqrt{\frac{(J-M+1)(J+M+1)(J-M+2)(J+M+2)}{(2J+1)(2J+3)^2(2J+5)}}\overline{N}\mathfrak{R}(0)\ ;$$

$\Delta J = 0$, $\Delta M = 0$:

$$\sqrt{\frac{2}{3}}\frac{J(J+1)-3M^2}{(2J-1)(2J+3)}\overline{N}\mathfrak{R}(0)\ ;$$

$\Delta J = -2$, $\Delta M = 0$:

$$\sqrt{\frac{3}{2}}\sqrt{\frac{(J-M)(J+M)(J-M-1)(J+M-1)}{(2J-3)(2J-1)^2(2J+1)}}\overline{N}\mathfrak{R}(0)\ ;$$

$\Delta J = 2$, $\Delta M = \pm 1$:

$$\mp\sqrt{\frac{(J-M+1)(J+M+1)(J\pm M+2)(J\pm M+3)}{(2J+1)(2J+3)^2(2J+5)}}\overline{N}\mathfrak{R}(\pm 1);$$

$\Delta J = 0$, $\Delta M = \pm 1$:

$$\pm\frac{(1\pm 2M)\sqrt{(J\pm M+1)(J\mp M)}}{(2J-1)(2J+3)}\overline{N}\mathfrak{R}(\pm 1)\ ; \quad (24)$$

$\Delta J = -2$, $\Delta M = \pm 1$:

$$\pm\sqrt{\frac{(J-M)(J+M)(J\mp M-1)(J\mp M-2)}{(2J-3)(2J-1)^2(2J+1)}}\overline{N}\mathfrak{R}(\pm 1)\ ;$$

$\Delta J = 2$, $\Delta M = \pm 2$:

$$\frac{1}{2}\sqrt{\frac{(J\pm M+1)(J\pm M+2)(J\pm M+3)(J\pm M+4)}{(2J+1)(2J+3)^2(2J+5)}}\overline{N}\mathfrak{R}(\pm 2)\ ;$$

$\Delta J = 0$, $\Delta M = \pm 2$:

$$-\sqrt{\frac{(J\pm M+1)(J\pm M+2)(J\mp M)(J\mp M-1)}{(2J-1)(2J+3)}}\overline{N}\mathfrak{R}(\pm 2)\ ;$$

$\Delta J = -2$, $\Delta M = \pm 2$:

$$\frac{1}{2}\sqrt{\frac{(J\mp M)(J\mp M-1)(J\mp M-2)(J\mp M-3)}{(2J-3)(2J-1)^2(2J+1)}}\overline{N}\mathfrak{R}(\pm 2)\ .$$

For all other $\Delta J$, $\Delta M$ $(a/\mathfrak{N}/b)_x$ is zero.

The strength of the transition between any two states, $|(a/\mathfrak{R}/b)_x|^2$, is simply the square of the corresponding coefficient of $\mathfrak{R}$ above, since the $\mathfrak{R}$'s are themselves normalized to unity.

The strength of a line (Eq. [9]) is easily obtained by noting that $\sum_i |(a_i/\mathfrak{R}/b_j)_x|^2$ must be independent of $j$; the probability of transition out of a state is independent of the quantum number $M$, so long as there is no external field to define a preferred direction in space. Then

$$S(A, B) = (2J_B + 1) \sum_i |(a_i/\mathfrak{R}/b_j)_x|^2 = (2J_A + 1) \sum_j |(a_i/\mathfrak{R}/b_j)_x|^2 . \quad (25)$$

Thus, one can arbitrarily choose $a_i$ with $M = 0$, say, and easily carry out the indicated summations. One finds

$$S(A, B) = \begin{cases} \overline{N}^2 \times \dfrac{(J+1)(J+2)}{2J+3}, & \Delta J = 2, \\ \overline{N}^2 \times \dfrac{2J(J+1)(2J+1)}{3(2J-1)(2J+3)}, & \Delta J = 0, \\ \overline{N}^2 \times \dfrac{J(J-1)}{2J-1}, & \Delta J = -2, \end{cases} \quad (26)$$

and finally

$$B(B, A) = \frac{4\pi^5\nu^2}{5c^3h^2} \overline{N}^2 \times \begin{cases} \dfrac{(J+1)(J+2)}{(2J+1)(2J+3)}, & \Delta J = 2; \\ \dfrac{2J(J+1)}{3(2J-1)(2J+3)}, & \Delta J = 0; \\ \dfrac{J(J-1)}{(2J-1)(2J+1)}, & \Delta J = -2. \end{cases} \quad (27)$$

It will be noted that equations (24) and (26) agree with Condon and Shortley's equations (6*a*) and (7), if one takes as the values of their $D$, $E$, $F$,

$$D = \frac{-\overline{N}}{(2J-1)(2J+3)},$$

$$E = 0,$$

$$F = \frac{\overline{N}}{(2J-1)[(2J-3)(2J+1)]^{1/2}}.$$

## 2. QUADRUPOLE MOMENT OF THE GROUND STATE OF $H_2$

For the discussion of the vibration-rotation spectrum of $H_2$ we are concerned simply with the quadrupole moment of $H_2$ in the ground state; in the notation of the preceding section we have $e = e'$. We shall first consider that part of the quadrupole moment that gives rise to radiation as an average over the electronic co-ordinates, depending on the nuclear separation $r$ as a parameter:

$$N(r) = 2e\left\{\frac{r^2}{4} + \overline{X_1^2}(r) - \overline{Z_1^2}(r)\right\}.$$

A later section will be devoted to finding the averages, $\overline{N}$, of this quantity over the various radial functions.

As a first approximation we have evaluated $N(r)$ on the basis of the Heitler-London wave function

$$F = [2\pi^2(1+S^2)]^{-1/2}[e^{-(r_{1a}+r_{2b})} + e^{-(r_{1b}+r_{2a})}],$$

where, in the usual notation, $r_{1a}$, $r_{1b}$, etc., are measured in Bohr radii ($a_H$), and $S$ is the overlap integral

$$S = \frac{1}{\pi}\int e^{-(r_{1a}+r_{1b})}d\vec{r}_1.$$

The integrations involved in computing $N(r)$ are elementary, and we obtain

$$\overline{Z_1^2} = \frac{1}{1+S^2}\left\{1 + \frac{r^2}{4} + S(e^{-r}/60)(60 + 60r + 27r^2 + 7r^3 + r^4)\right\};$$

$$\overline{X_1^2} = \frac{1}{1+S^2}\{1 + S(e^{-r}/15)(15 + 15r + 6r^2 + r^3)\};$$

$$S = e^{-r}\left(1 + r + \frac{r^2}{3}\right);$$

$$N(r) = \tfrac{2}{5}er^2\frac{S^2}{1+S^2}.$$

Values of $N(r)$ for the range of $r$ which is of interest appear in Table 3. From $r = 1a_H$ to $r = 2a_H$, the function is represented to within 1 per cent by

$$N(r) = e\{0.285 + 0.263(r - 1.4) - 0.084(r - 1.4)^2\}.$$

The effective quadrupole moment thus appears as a positive definite quantity, which arises as the difference between a positive contribution $er^2/4$ from each proton and a negative definite contribution $-e(\overline{Z_1^2} - \overline{X_1^2})$ from each electron. The cancellation is fairly severe, the whole being of the order of one-fourth of the positive and negative components separately. This is but one element of the situation that makes it impossible to accept the foregoing results as reliable. The Heitler-London function is known to be a rather poor approximation to the correct function for this molecule, and in particular it gives too great a probability density of electrons far from the nuclei and also near the nuclear axis beyond the nuclei.[4] Thus, the function will be expected to give too large a value for $\overline{Z_1^2} - \overline{X_1^2}$, and accordingly too small a value of $N(r)$. Furthermore, it is the rate of change of $N(r)$, and the derivatives of this quantity, which finally determine the intensity of radiation; since the Heitler-London function becomes worse in a systematic way as the nuclei approach, one will expect a systematic change in the error of $N(r)$, and consequently a large error in the derivatives. For these reasons we have con-

[4] H. M. James and A. S. Coolidge, *J. Chem. Phys.*, **1**, 825, 1933. Cf. also H. M. James, *ibid.*, **3**, 9, 1935.

sidered it necessary to compute $N(r)$ by a more accurate method. We present the Heitler-London results merely for comparison.

For the more accurate computation we have employed the method which we have previously[4] devised for a careful study of the ground state of $H_2$. The function is of the form

$$F = \sum_{mnjkp} C_{mnjkp}[mnjkp] ,$$

where $[mnjkp]$ stands for $(\frac{1}{2\pi})e^{-\delta(\lambda_1+\lambda_2)}\,(\lambda_1^m\lambda_2^n\mu_1^j\mu_2^k\rho^p + \lambda_1^n\lambda_2^m\mu_1^k\mu_2^j\rho^p)$, with

$$\lambda_1 = \frac{r_{1a} + r_{1b}}{r} , \qquad \mu_1 = \frac{r_{1a} - r_{1b}}{r} , \qquad \rho = \frac{2r_{12}}{r} .$$

The manipulation of these functions is fully described in the paper referred to, and the coefficients $C_{mnjkp}$ are there given for some quite

TABLE 1

COEFFICIENTS IN WAVE FUNCTIONS

| Terms | $\delta=0.75$ | | | $\delta=0.875$ | | |
|---|---|---|---|---|---|---|
| | $r=1.1a_H$ | $r=1.3$ | $r=1.5$ | $r=1.5$ | $r=1.7$ | $r=1.9$ |
| [00000]..... | 2.61623 | 2.46597 | 2.03888 | 2.37301 | 2.10568 | 1.76079 |
| [00020]..... | 0.86493 | 0.75604 | 0.64527 | 1.23006 | 1.07702 | 0.93081 |
| [00110]..... | −0.22738 | −0.26338 | −0.26011 | −0.42616 | −0.42101 | −0.39320 |
| [10000]..... | −0.30175 | −0.59461 | −0.59251 | −0.59444 | −0.66606 | −0.61730 |
| [00001]..... | 0.41034 | 0.25671 | 0.16236 | 0.33075 | 0.22241 | 0.15108 |

accurate functions, for values of $r$ ranging from 1.2 $a_H$ to 1.7 $a_H$. For the present purpose it was sufficient to use simpler, though less accurate, functions, such as the five-term function containing the terms which we had already found were sufficient, at $r = 1.4$, to give a very fair approximation to the correct energy value. Functions were accordingly worked out for five different values of $r$, using two different choices of $\delta$. Table 1 shows the coefficients defining the normalized functions, and Table 2 gives the corresponding calculated binding energy, compared with a Morse curve fitted to the experi-

mental data of Beutler and Jünger[5] and of Jeppesen.[6] It will appear from these results that, for a given $r$, the best value of $\delta$ is somewhat greater for the five-term function than it is when more terms are employed. The two independent computations at $r = 1.5$ make possible a check on the effect upon $N(r)$ of errors in the wave function.

TABLE 2

COMPUTED BINDING ENERGY, IN ENERGY VALUE

| | $\delta=0.75$ | | | $\delta=0.875$ | | |
|---|---|---|---|---|---|---|
| | $r=1.1$ | $r=1.3$ | $r=1.5$ | $r=1.5$ | $r=1.7$ | $r=1.9$ |
| Variational function | 3.95 | 4.49 | 4.37 | 4.55 | 4.19 | 3.61 |
| Morse curve | 4.08 | 4.66 | 4.67 | 4.67 | 4.38 | 3.94 |

$N(r)$ may be evaluated in terms of the set of quantities employed in computing the energy of the functions. Since $\overline{X_1^2} = \overline{Y_1^2} = \overline{X_2^2}$, we may write

$$N(r) = e\left\{\frac{r^2}{2} + \tfrac{1}{2}(\overline{X_1^2 + Y_1^2}) + \tfrac{1}{2}(\overline{X_2^2 + Y_2^2}) - \overline{Z_1^2} - \overline{Z_2^2}\right\}.$$

Now, in elliptical co-ordinates we have

$$X_1^2 + Y_1^2 = \tfrac{1}{4}r^2(\lambda_1^2 - 1)(1 - \mu_1^2);\ Z_1^2 = \tfrac{1}{4}r^2\lambda_1^2\mu_1^2 .$$

Since all the terms in the wave function involve the electrons symmetrically, we have relations such as $(\lambda_1^2 + \lambda_2^2)[mnjkp] = [(m+2)njkp] + [m(n+2)jkp]$, and therefore

$$\begin{aligned} N(r) = (er^2/8)\sum_{mnjkp}\sum_{m'n'j'k'p'} C_{mnjkp}C_{m'n'j'k'p'}[&S(m+2, n, j, k, p; m', n', j', k', p') \\ &+ S(m, n+2, j, k, p; m', n', j', k', p') \\ &+ S(m, n, j+2, k, p; m', n', j', k', p') \\ &+ S(m, n, j, k+2, p; m', n', j', k', p') \\ &- 3S(m+2, n, j+2, k, p; m', n', j', k', p') \\ &- 3S(m, n+2, j, k+2, p; m', n', j', k', p') \\ &+ 2S(m, n, j, k, p; m', n', j', k', p')] . \end{aligned}$$

[5] *Zs. f. Phys.*, **101**, 304, 1936. [6] *Phys. Rev.*, **44**, 165, 1933.

$S$ is here the integral $\int [m,n,j,k,p][m',n',j',k',p']\,d\tau_1 d\tau_2$, which may be evaluated by the methods given in reference 4.

The results of these computations appear in Table 3. They can be approximated by

$$N(r) = e\{0.442 + 0.307(r - 1.4) - 0.205(r - 1.4)^2\},$$

the values of which are also given in Table 3. It is reassuring that no serious discrepancy appears between the two calculations at $r = 1.5$. Since the values of $\delta$ used were, respectively, considerably smaller

TABLE 3

EFFECTIVE QUADRUPOLE MOMENT $N(r)$, IN PROTON CHARGE × (BOHR RADIUS)$^2$

| | $r=1.1a_H$ | $r=1.3$ | $r=1.5$ | $r=1.7$ | $r=1.9$ | $r=4.5$ |
|---|---|---|---|---|---|---|
| Computed, $\delta=0.75$ | 0.331 | 0.398 | 0.462 | ..... | ..... | ..... |
| Computed, $\delta=0.875$ | ..... | ..... | .478 | 0.515 | 0.544 | ..... |
| Computed, $H-L$ function | .198 | .256 | .310 | .358 | .395 | 0.147 |
| Quadratic approximation | .331 | .409 | .470 | .515 | .544 | − .576 |
| Cubic approximation | 0.331 | 0.409 | 0.470 | 0.515 | 0.544 | 0.178 |

and (probably) somewhat larger than the most advantageous choice, the errors in the functions are presumably of opposite character, particularly with respect to the scale of the electronic distribution as a whole. An intermediate value for $N(r)$, such as that given by the quadratic approximation, is therefore probably quite near the truth. We estimate the error as of the order of 3 per cent in $N(r)$, as given by that expression, 5 per cent in $dN/dr$, and 20 per cent in $d^2N/dr^2$. It will be noted that, as expected, the value computed with the Heitler-London function is too low, though the rate of change is about right. On the other hand, the second derivative, which is particularly interesting for our present purpose, comes out too small by more than half.

In view of the importance of the second derivative, it is well to have some check on the order of magnitude of the third. This can be obtained by considering the situation for large values of $r$, where

the Heitler-London function is less in error. Thus, the cubic approximation

$$N(r) = e\{0.442 + 0.303(r - 1.4) - 0.212(r - 1.4)^2 + 0.028(r - 1.4)^3\}$$

is indistinguishable from the quadratic expression over the range of interest but is much better at large $r$, as will be seen from Table 3.

## 3. EVALUATION OF THE RADIAL INTEGRALS

In order to discuss the intensities of the fundamental rotation-vibration band, as well as the first two overtone bands, we have now to consider the evaluation of the radial integrals

$$\overline{N}(e, v, J; e', v', J') = \int R_{evJ}(r)N(r)R_{e'v'J'}(r)dr$$

for $v = 0$, $v' = 1$, 2, and 3. These integrals will depend principally on $v'$, and also to a minor extent on the values of $J$ and $J'$. Following Gibson, Rice, and Bayliss,[7] we shall neglect the dependence on rotational quantum numbers, though it should be noted that, in the present problem, the values of the integrals are probably more sensitive to changes in the effective potential curve than in the case which they discuss. We shall, then, use radial functions $R_{evJ}$, $R_{e'v'J'}$, computed for $J = J' = 0$.

It is instructive to regard $N(r)$ as expanded in powers of the displacement from the potential minimum $r_e$, and to investigate the components of $\overline{N}$ produced by the separate terms. Thus, if

$$N(r) = N_e + N'_e(r - r_e) + \tfrac{1}{2}N''_e(r - r_e)^2 + \ldots\ldots$$

we have

$$\overline{N}_{v, v'} = N'_e\overline{(r - r_e)}_{v, v'} + \tfrac{1}{2}N''_e\overline{(r - r_e)^2}_{v, v'} + \ldots\ldots$$

For any potential curve, the contribution from the constant term $N_e$ is, of course, zero, because different wave functions are orthogonal. The contributions from the higher terms are quite markedly

[7] *Ibid.*, **44**, 193, 1933.

dependent upon the exact form of the potential curve. The simplest possible approximation is a parabola adjusted to give the proper values of the spectroscopic constants $r_e$ and $\omega_e$, and it is a property of the familiar wave functions of the harmonic oscillator that $\overline{(r - r_e)^n_{0,v'}}$ vanishes unless $n - v'$ is even and positive. Thus, to the approximation in which the third and higher derivatives of $N$ can be neglected, the strength of the fundamental band would be determined entirely by $dN/dr$ evaluated at $r_e$, the first overtone band by $d^2N/dr^2$, with other overtones absent. Explicitly,

$$\overline{(r - r_e)}_{0,1} = -(Rm/\omega_e\mu)^{1/2}a_H ,$$

$$\overline{(r - r_e)^2}_{0,2} = \sqrt{2}\,\frac{Rm}{\omega_e\mu}\,a_H^2 ,$$

$$\overline{(r - r_e)^3}_{0,3} = -\sqrt{6}(Rm/\omega_e\mu)^{3/2}a_H^3 ,$$

$$\overline{(r - r_e)^3}_{0,1} = -3(Rm/\omega_e\mu)^{3/2}a_H^3 ,$$

with $R$ the Rydberg constant, $m$ the mass of the electron, and $\mu$ the reduced mass of the nuclei. In this and in ensuing expressions, the signs correspond to wave functions chosen so as to have the first loop for small $r$ positive.

When a better approximation to the true potential curve is used, its departure from the parabolic form results in contributions from each derivative for every value of $v'$. For the Morse curve, general expressions for $\overline{(r - r_e)^n}_{0,v}$ may readily be found by extending the treatment of Rosenthal.[8] We introduce the functions

$$F_n(x) = \frac{\dfrac{d^n\Gamma(x)}{dx^n}}{\Gamma(x)} ,$$

with $F_0(x) = 1$, $F_1(x) = \psi(x)$, $F_2(x) = \psi'(x) + [\psi(x)]^2$, $F_3(x) = \psi''(x) + 3\psi(x)\psi'(x) + [\psi(x)]^3$. By repeated differentiation of the relation $\Gamma(x) = (x - 1)\Gamma(x - 1)$, we readily obtain

$$(x - 1)[F_n(x) - F_n(x - 1)] = nF_{n-1}(x - 1) .$$

[8] *Proc. Nat. Acad.*, **21**, 281, 1935.

The final formulae are found to be

$$\overline{(r-r_e)^n_{0,1}} = n\left\{\left[\frac{(-1)^n(k-3)^{1/2}}{a^n(k-2)}\right]F_{n-1}(k-2) + ((\ln k)/a)\overline{(r-r_e)^{n-1}_{0,1}}\right.$$
$$- \tfrac{1}{2}(n-1)((\ln k)/a)^2\overline{(r-r_e)^{n-2}_{0,1}}$$
$$\left. + \tfrac{1}{2}\cdot\tfrac{1}{3}\cdot(n-1)(n-2)((\ln k)/a)^3\overline{(r-r_e)^{n-3}_{0,1}} - \ldots.\right\}$$

$$\overline{(r-r_e)^n_{0,2}} = n\left\{\left[\frac{(-1)^n(k-5)^{1/2}}{a^n(k-3)(2!(k-2))^{1/2}}\right][F_{n-1}(k-3)\right.$$
$$+ (n-1)F_{n-2}(k-3)] + ((\ln k)/a)\overline{(r-r_e)^{n-1}_{0,2}}$$
$$\left. - \tfrac{1}{2}(n-1)((\ln k)/a)^2\overline{(r-r_e)^{n-2}_{0,2}} + \ldots.\right\}$$

$$\overline{(r-r_e)^n_{0,3}} = n\left\{\left[\frac{(-1)^n(k-7)^{1/2}}{a^n(k-4)(3!(k-2)(k-3))^{1/2}}\right][2F_{n-1}(k-4)\right.$$
$$+ 3(n-1)F_{n-2}(k-4) + (n-1)(n-2)F_{n-3}(k-4)]$$
$$\left. + ((\ln k)/a)\overline{(r-r_e)^{n-1}_{0,3}} - \tfrac{1}{2}(n-1)((\ln k)/a)^2\overline{(r-r_e)^{n-2}_{0,3}} + \ldots.\right\}.$$

Here, $a$ and $k$ are a reciprocal length and a number characteristic of the Morse potential function. When the curve is adjusted to give the correct value of the spectroscopic constant $\omega_e x_e$ expressed in wave numbers, we have

$$a = (\mu\omega_e x_e/mR)^{1/2}a_H^{-1}\,; \qquad k = \frac{\omega_e}{\omega_e x_e}\,.$$

Using the spectrocopic constants given by Jeppesen,[6] we obtain the numerical values given in Table 4, where they may be compared with the results given by the parabolic approximation. It will be noticed that the terms arising from the nonparabolic form are by no means unimportant.

Table 4 also gives the values of $N(r)$ averaged over both parabolic and Morse curve functions and computed both from the quadratic and the cubic approximations. For each potential function,

the effect of including the cubic term is quite unimportant; and it is clear that, so far as the first three transitions are concerned, we have calculated $N(r)$ over a sufficient range to determine the results with fair accuracy. On the other hand, in going from the parabola to the Morse curve, we encounter major changes in the ratios of the $\overline{N}$'s, and still greater changes in the ratios of the predicted band intensi-

TABLE 4

VALUES OF INTEGRALS OVER WAVE FUNCTIONS BELONGING TO VARIOUS APPROXIMATE POTENTIAL CURVES

| | Parabola | | Morse | | Pöschl-Teller |
|---|---|---|---|---|---|
| $\overline{(r-r_e)}_{0,1}$ | −0.1644 | | −0.1668 | | |
| $\overline{(r-r_e)}_{0,2}$ | 0 | | − .0209 | | |
| $\overline{(r-r_e)}_{0,3}$ | 0 | | − .0044 | | |
| $\overline{(r-r_e)^2}_{0,1}$ | 0 | | − .0246 | | |
| $\overline{(r-r_e)^2}_{0,2}$ | 0.0382 | | .0355 | | |
| $\overline{(r-r_e)^2}_{0,3}$ | 0 | | .0113 | | |
| $\overline{(r-r_e)^3}_{0,1}$ | −0.0133 | | − .019 | | |
| $\overline{(r-r_e)^3}_{0,2}$ | 0 | | .009 | | |
| $\overline{(r-r_e)^3}_{0,3}$ | −0.0109 | | −0.007 | | |
| | Quad. Appr. | Cubic Appr. | Quad. Appr. | Cubic Appr. | Quad. Appr. |
| $\overline{N}_{0,1}/e$ | −0.0505 | −0.0502 | −0.0462 | −0.0459 | −0.0441 |
| $\overline{N}_{0,2}/e$ | −0.0078 | − .0081 | − .0137 | − .0136 | − .0166 |
| $\overline{N}_{0,3}/e$ | 0 | −0.0003 | −0.0037 | −0.0039 | −0.0044 |

ties, which are proportional to the squares of the $\overline{N}$'s. Analysis of the situation shows that the differences are due mainly to the asymmetry of the Morse curve, which tends to increase the relative intensities of the overtone bands. In view of the sensitiveness of the results to the asymmetry of the potential function, and the fact that the Morse curve is in this respect by no means a good approximation to the true curve, it seemed important to examine the results obtained with some more accurate potential function.

The most direct indication of the asymmetry of any potential

curve is the calculated value of the spectroscopic constant $a_e$, which determines the principal term in the energy due to the interaction of rotation and vibration, and is proportional to the rate of change of the average moment of inertia of the molecule with the vibrational quantum number. According to Jeppesen,[6] $a_e$ has for $H_2$ the unusually large value 3.07; whereas the parabola gives $a_e = -6B_e^2/\omega_e = -5.03$, and the Morse curve, as here constructed, gives, according to Pekeris,[9] $a_e = 6x_eB_e[(B_e/x_e\omega_e)^{1/2} - B_e/x_e\omega_e] = 2.36$. Now, the curve given by Pöschl and Teller[10] contains four adjustable parameters, one more than the Morse curve. It can therefore be adjusted to give the correct value of $a_e$, as well as $B_e$, $\omega_e$, and $x_e\omega_e$, provided one does not require it to give the proper dissociation energy; this proviso (which also applies to the Morse curve used) is legitimate in the present case, since it affects only the more remote parts of the curve, which are practically without influence on the first few wave functions. We have worked out such a curve, but postpone for a later communication a discussion of the method. The equation for the curve is

$$\frac{V(r)}{hc} = R_\mu^m a^2 \left[ \frac{\nu(\nu - 1)}{\sinh^2 a(r - r_0)} - \frac{\mu(\mu + 1)}{\cosh^2 a(r - r_0)} \right],$$

with $a = 0.52414 a_H^{-1}$, $r_0 = -0.0696 a_H$, $\nu = 24.773$, $\mu = 57.436$. (The $\mu$ in the first factor is, of course, the reduced mass of the molecule, as before.)

It is possible to obtain analytical expressions for $\overline{(r - r_e)^n}_{v,v'}$, using the wave functions of the Pöschl-Teller curve; but the results appear as a series converging too slowly for practical computation. We have been fortunate in being able to use the differential analyzer of the Massachusetts Institute of Technology in evaluating the integrals $\overline{N}_{v,v'}$. The technique of this work need not be described here, as it was very similar to that previously discussed in connection with another investigation.[11] Unfortunately, the curve employed

[9] *Phys. Rev.*, **45**, 98, 1934.

[10] *Zs. f. Phys.*, **83**, 143, 1933.

[11] Coolidge, James, and R. D. Present, *J. Chem. Phys.*, **4**, 193, 1936.

was not exactly that just given, but one based upon Lotmar's[12] formulae for constructing a Pöschl-Teller curve so as to reproduce given spectroscopic constants. These formulae contain an error which was not detected until after the work had been completed, so that the results obtained correspond to a potential curve giving spurious spectroscopic constants, slightly different from the true values. We have made an estimate of the corresponding errors in the $\overline{N}$'s by comparing the values computed with Morse curves so constructed as to give, respectively, the spurious and the true spectroscopic constants. The appropriate corrections (amounting to 5 per cent for $\overline{N}_{0,1}$ and less for $\overline{N}_{0,2}$ and $\overline{N}_{0,3}$) have been applied to the values given in Table 4. We believe these may be relied upon to 1 or 2 per cent, except for $\overline{N}_{0,3}$, which may be in error by as much as 5 per cent, since the extensive cancellation between contributions to the integral from alternating positive and negative parts of the wave function for $v = 3$ makes the final result very sensitive to errors in the form of the potential curve used and in the operation of the machine. It is upon these values of $\overline{N}$ that the subsequent discussion will be based. It is to be regretted that we did not realize the desirability of obtaining the separate values of the $\overline{(r - r_e)^n}_{0,v}$ for comparison with the parabola and Morse results. Greater accuracy (as far as machine operation is concerned) could have been secured thus, by eliminating the contribution from $n = 0$ at the outset instead of relying upon the machine to cancel the corresponding term in the integral.

## 4. RESULTS AND DISCUSSION

Evaluating the constants of equation (27), using the results of the previous section, we have

$$B(B, A) = C_{v,v'}\sigma^2 \times \begin{cases} \dfrac{(J+1)(J+2)}{(2J+1)(2J+3)}, & \Delta J = 2, \\ \dfrac{2J(J+1)}{3(2J-1)(2J+3)}, & \Delta J = 0, \\ \dfrac{J(J-1)}{(2J-1)(2J+1)}, & \Delta J = -2, \end{cases}$$

[12] *Zs. f. Phys.*, **93**, 528, 1935.

with

$$C_{0,1} = 6.55 \times 10^{-11},$$
$$C_{0,2} = 0.925 \times 10^{-11},$$
$$C_{0,3} = 0.067 \times 10^{-11}.$$

Special values of $B(B, A)$ for the first line of the $S$ branch in each band are as follows:

| | |
|---|---|
| Fundamental | $8.84 \times 10^{-4}$, |
| First overtone | $4.35 \times 10^{-4}$, |
| Second overtone | $0.65 \times 10^{-4}$. |

With these we may compare the value $B = 1.71 \times 10^{5}$ given by Bourgin[13] for the first line of the $R$ branch of the fundamental band of *HCl*. (The number given by Bourgin is actually greater by a factor of $c$, since he defines $B(B, A)$ in terms of energy density per unit difference of frequency rather than of wave number.)

As the proper measure of the intensity of an absorption line, we take the integrated absorption coefficient $\int \alpha(\sigma)d\sigma$. It is well known[14] that if one neglects the width of the line, which is, of course, extremely small for the very improbable transitions which we are considering, then

$$\int \alpha(\sigma)d\sigma = (h\nu/c)B(B, A)N(B).$$

For a given density of absorbing atoms, the integrated absorption coefficients for the $H_2$ lines thus stand in the ratios $8.1 \times 10^{-9}$, $7.3 \times 10^{-9}$, and $1.5 \times 10^{-9}$, respectively, to that for the *HCl* line.

The ratios of the intensities of the first and second overtones to the fundamental are 0.91 and 0.19, respectively. These large values are in striking contrast to the corresponding values, 0.016 and $\sim$ 0.001, for *HCl*.[15] This fact can be accounted for as follows: The expression for $\int \alpha(\sigma)d\sigma$ is proportional to $\nu^3$ for quadrupole radiation and to $\nu$ for dipole radiation, which increases the ratio of the first

[13] *Phys. Rev.*, **29**, 794, 1927.

[14] Condon and Shortley, *op. cit.*, pp. 109 ff.

[15] J. L. Dunham, *Phys. Rev.*, **34**, 438, 1929.

overtone to the fundamental by a factor of 4. A further factor of 4 in this ratio comes from the fact that in our case (to judge by the Morse curve results) $N_e'\overline{(r-r_e)}_{0,2}$ and $\frac{1}{2}N_e''\overline{(r-r_e)^2}_{0,2}$ make comparable contributions to $\overline{N}_{0,2}$, whereas Rosenthal[8] finds that for *HCl* the term analogous to the second is probably negligible because of the smallness of $P_e''$ (the value at the potential minimum of the second derivative of the electric moment function $P[r]$). The remaining factor of 3 is due to the exceptional asymmetry of the potential curve for $H_2$, for which $a/B_e = 0.052$, as against 0.029 for *HCl*. Of these three factors, the first is certainly, and the second possibly, a general characteristic of quadrupole radiation, so that it is to be expected that in quadrupole bands in general the overtones will have greater intensities, relative to the fundamentals, than in dipole bands, although not to the exceptional degree shown in $H_2$.

In conclusion, we wish again to express our indebtedness to the Electrical Engineering Department of the Massachusetts Institute of Technology for the use of the differential analyzer, and to thank Dr. E. L. Vernon for assistance in its operation, as well as in some of the numerical work of the paper. In addition, one of us (H. M. J.), wishes to acknowledge gratefully a grant from the Society of Sigma Xi, which materially aided in the progress of the work.

PURDUE UNIVERSITY AND HARVARD UNIVERSITY

# ON THE DETERMINATION OF COLOR INDICES OF STARS FROM A CLASSIFICATION OF THEIR SPECTRA

W. W. MORGAN

ABSTRACT

A discussion of Bottlinger's photoelectric color indices of the near-by stars of types $F_5$–Mo is made in combination with a classification of spectral type and luminosity. It is found that, within the errors of observation, the intrinsic colors of the stars are uniquely defined by their spectral types and absolute magnitudes.

The color indices of stars unaffected by selective interstellar absorption can be predicted with accuracy from a classification depending on line ratios within their spectra. The suitability of the $F_5$–Go main sequence stars as color standards is pointed out. For these stars color indices can be predicted which have a mean error of the order of $\pm 0.02$ mag. on Bottlinger's scale.

1. In an important paper published some years ago, Öpik[1] investigated the problem of the practical determination of luminosities of stars from their spectral types and color indices. Elaborating on the pioneer work of Adams, Seares, Lindblad, and others, he found a good correlation between absolute magnitude and color index for stars of the same spectral type, except in the case of the A and early F stars. For types G and K, especially, there is a considerable increase in color index on passing from the dwarfs to the normal giants and on to the supergiants.

Investigations of the nature and amount of reddening due to interstellar causes which have been made since Öpik's paper was published, have shown that in the case of B and O stars a considerable part of the reddening which is observed with increasing luminosity is due to interstellar material. If stars of any given apparent magnitude are considered, then the more luminous ones are also the more distant and will be more likely to be affected by interstellar reddening. A marked effect is to be expected particularly in the case of the B and O stars which possess the characteristics of high luminosity and a high degree of concentration toward the galactic plane. In the case of the G and K stars, however, the reddening observed with increasing luminosity is largely intrinsic, as it is well shown in ob-

[1] *Tartu Pub.*, **27**, 1, 1929.

jects which are so near to the sun that the reddening cannot be accounted for by interstellar causes.

It is the purpose of the present paper to reverse the problem treated by Öpik and to investigate the accuracy with which intrinsic color indices (that is, color indices which are unaffected by selective interstellar absorption) are defined by spectral type and luminosity for stars of types F5–K5. Instead of using explicit values of the absolute magnitudes of the stars discussed, however, an abstract luminosity classification based on spectroscopic criteria will be employed. The advantages in the use of such a classification have been discussed in a recent paper.[2]

2. The stars suitable for discussion are limited by considerations of selective interstellar absorption. If observed color indices are to be considered essentially intrinsic in nature, then it is necessary to include only the near-by stars. A correction to the observed color index of more distant objects as a simple function of the distance is not reliable because of the marked irregularity in the distribution of the absorbing material; in addition, the characteristics of the interstellar matter are not, as yet, completely known. In a few regions of the northern sky interstellar absorption becomes appreciable at distances which are probably somewhat less than one hundred parsecs. If we limit ourselves to stars whose apparent distances are less than one hundred parsecs, however, selective effects should, in general, be quite small because of the large ratio of selective to total absorption. A round figure of HR magnitude 4.00 has been set as the faint limit for ordinary giants of types F5–K5. The apparent distance of very few objects so chosen will be as great as 100 parsecs. A fainter limit has been set for the main-sequence stars; the ones included all lie within a distance of 50 parsecs.

Because of the low space density and the great distances of the supergiants, it is necessary to include more distant objects if a representative selection is to be made. The limiting apparent visual magnitude of 4.00 has been adopted. This is the same limit which was set for the normal giants; as a result, the mean distance of the supergiants will be much greater, and the observed color indices are more likely to be affected by interstellar reddening. As some of the bright

[2] *Ap. J.*, **85**, 380, 1937.

supergiants are located at apparent distances of around 500 parsecs, their colors may be appreciably affected by selective absorption. The majority of the F, G, and K supergiants brighter than the limit set are, however, located in regions which are relatively transparent up to distances of 300–400 parsecs.

3. The observational material is of two kinds: accurate color indices and accurate classifications both of spectral type and luminosity. Their low accidental error, together with their freedom from several troublesome sources of systematic error, makes photoelectric color indices most suitable for discussion. The series of photoelectric color indices of Bottlinger[3] will be used in the present discussion.[4] Bottlinger's original measures have been used in preference to his reduction of the color indices to King's scale because of the complicated nature of the relation between the two color systems. The scale of the photoelectric color indices is approximately one-half that of King's; the average mean error of a single observation of color of highest weight is $\pm 0.016$ mag. on Bottlinger's scale. Observations of full weight only have been used. All stars known to have composite spectra have been omitted.

In spite of the fact that all the stars used in the discussion are bright and are included in two or more series of spectral classification, there are reasons for re-examining the question of their classification. In problems which make use of spectral types of faint stars the assumption is usually made that a star of the tenth magnitude, for example, which has been classified as K0, is on the system of the bright K0 stars. This, in turn, is dependent on the use of the same spectral criteria for bright stars and for faint ones, and to a certain extent on the comparability of the spectral dispersions. In general, the bright stars are observed with rather high dispersion to obtain high accuracy in classification, while low dispersion is employed of necessity for faint objects. In the latter case the accidental errors

[3] *Veröff. Berlin-Babelsberg*, **3**, Part IV, 1923.

[4] The carefully compiled catalogue of Bottlinger's re-reduced color indices and the new determinations of W. Becker (*Veröff. Berlin-Babelsberg*, **10**, Part 3, 1933) indicate the same conclusions found in the present paper from Bottlinger's earlier published measures. The excellent infrared color indices of Hall (*Ap. J.*, **79**, 145, 1934) have not been used because of their great difference in effective wave length as compared with ordinary photographic and visual systems.

of classification tend to be higher, and the connection to the system of the brighter stars is not as well defined as it would be if the fainter stars could be observed with higher dispersion.

From these considerations certain general characteristics of an "optimum" classification for bright and faint stars may be described: (*a*) The dispersion should be low for photographic efficiency. (*b*) The resolving-power of the spectrograms should be high for accuracy in classification and for the determination of spectroscopic luminosities. (*c*) There should be no systematic difference in the quality of the spectrograms which depends on the apparent magnitude of the stars. The use of short-focus cameras on slit spectrographs satisfies all of the conditions. Short-focus lenses of the type designed by G. W. Moffitt, when used in connection with the modern high-speed, fine-grain emulsions, give excellent low-dispersion plates of high resolving-power which are suitable for the determination both of spectral types and of spectroscopic absolute magnitudes. If care is taken that spectrograms of faint stars are not systematically underexposed in comparison with bright standard stars, the possibility of serious systematic error as a function of apparent magnitude is eliminated. The great disadvantage of the slit spectrograph is, of course, that only one star can be observed at a time. For this reason, where very large numbers of stars are to be observed, the objective prism method is the only one to be considered.

The spectra of all the stars included in the present discussion have been obtained with a short-focus Moffitt camera attached to the one-prism Bruce spectrograph on the forty-inch telescope. Criteria of the greatest sensitivity to changes in spectral type were selected, and a system was set up which is based on the Draper types of stars brighter than visual magnitude 4.0. The system is in close agreement with the Draper system; there is a maximum correction of about two-tenths of a spectral type at G5. The procedure adopted at Mount Wilson of placing type Mo immediately after K5 in the case of the giants has been followed; spectroscopic evidence and that of the color indices agree in making the interval between K5 and Mo no larger than that between K4 and K5.

The criteria used for the classification of spectral type are listed in Table 1.

The spectral type at which the components of line ratios become equal probably varies considerably with the dispersion used. For this reason the system of classification should be considered as being defined by the whole catalogue of stars which have been classified, rather than by arbitrary measures of certain spectral criteria.

In addition to the classification of spectral type, the stars have been divided into a number of luminosity classes by means of several well-known line ratios which have been employed for the determina-

TABLE 1

CRITERIA OF SPECTRAL TYPE

| Type | Criteria | Remarks |
|---|---|---|
| F5..... | 4045/*Hδ* | As in α Persei |
| G0..... | 4045/*Hδ*<br>*Hδ*>4226 | 4045=H*δ* at G0 on spectrograms of the dispersion used |
| G3..... | 4045>*Hδ*<br>*Hδ*=4226* | |
| G4–K0..... | 4226 is interpolated between 4045 and *Hδ* | |
| K1–K5..... | 4045/4226 | 4045=4226 at K1 on spectrograms of the dispersion used |
| M0..... | *Ti O* 6159 | Red *Ti O* band suspected at K5, faintly present at M0 |

* On spectrograms of the dispersion used.

tion of absolute magnitudes. These ratios are listed in Table 2. The luminosity classes used are defined in Table 3.

A catalogue of the spectral and the luminosity classification of 114 stars is given in Table 4. All stars of types F5–M0 brighter than 4.00 in the *Revised Harvard Photometry* for which Bottlinger measured color indices are included; in addition, a number of fainter main-sequence stars have been added. The columns of the table give: (1) the serial number; (2) the name of the star; (3) the position for 1900; (4) the HR visual magnitude and Draper type; (5) the Yerkes spectral type, with the number of spectrograms on which it was determined in parentheses; (6) the total range in the individual determinations of the spectral type; (7) the luminosity class; (8) the

photoelectric color index as measured by Bottlinger, with the number of first-class observations on which it depends.

The mean error of a determination of spectral type from one plate was computed by Schlesinger's[5] method for all stars for which there were between three and six plates. The accuracy varies considerably with spectral type; the highest precision is obtained for stars of types K1–M0, where half-tenths were commonly estimated. Next in order of accuracy are the F4–G0 stars, while the classification of

TABLE 2

LINE RATIOS USED FOR LUMINOSITY CLASSIFICATION

| Line Ratio | Spectral Range Used | Remarks |
|---|---|---|
| 4045/4077 | F5–M0 | Fundamental criterion of luminosity over great range in spectral type |
| 4226/4171 | F5–G0 | Especially useful in detecting F-type supergiants |
| 4260/4215 | F5–M0 | Lines weak in F stars for dispersion used; especially useful in separating G8–K1 subgiants from dwarfs |
| 4325/4290 | F5–M0 | Rather weak in F stars for dispersion used; useful in detecting supergiants of types G5–K5 |
| 4071/$H\delta$ | G5–M0 | Useful in detecting supergiants of types G5–K5 |

G1–G9 spectra has the lowest accuracy of all. The values of the mean error for individual spectrograms are given in units of tenths of a spectral class.

| | F4–G0 | G1–G9 | K0–K5 |
|---|---|---|---|
| Mean error of one plate | ±0.8 | ±1.3 | ±0.4 |

The mean catalogue values have been rounded off to the nearest half-tenth; systematic errors due to changes in the personality of the observer and the relatively indefinite nature of the definition of the classification system tend to make the external errors some-

[5] *A.J.*, **46**, 162, 1937.

TABLE 3

LUMINOSITY CLASSES

| Class | Description | Type Stars | | $M_{vis}$ | Source of $M$ and Remarks |
|---|---|---|---|---|---|
| *Ia* | Supergiants of types F4–F8 of highest luminosity | F4 | $\epsilon$ Aur | −7.2 | From Kuiper's solution of orbit* |
| | | F7 | $\rho$ Cas | | |
| | | F7+ | $\delta$ CMa† | | |
| *Ib* | Supergiants of types F4–F8 of somewhat lower luminosity | F5 | $\alpha$ Per | −4.4 | In Perseus cluster: cluster modulus = $6^{M}3$§ |
| | | F7 | $\alpha$ UMi | −4.2 | From c.p.m. companion, which is main-sequence F2 star. Absolute magnitude of companion assumed to be +2.5. Mean error is equal to dispersion of F2 dwarfs, which is around ±1 mag. |
| I | Supergiants of types later than F8 | G0 | $\zeta$ Mon | | |
| | | G4 | 9 Peg | | |
| | | G7+ | $\epsilon$ Gem | | |
| | | K1+ | $\epsilon$ Peg | | |
| | | K2 | $\eta$ Per | −4.8 | From c.p.m. companion, which is main-sequence B9 star. Absolute magnitude of companion is considered 0.0 with a mean error equal to the dispersion of B9 dwarfs, which is around ±1 mag. |
| | | K3+ | $\xi$ Cyg | | |
| | | K5+ | $\alpha$ Sco | −4.0‡ | From parallax of diffuse nebulosity in which Scorpius stars are involved |
| | | (M1) | $\alpha$ Ori | | |
| II | Stars intermediate in luminosity between supergiants and ordinary giants | G1 | $\epsilon$ Leo | | |
| | | G9 | $\zeta$ Cyg | | |
| | | K2+ | $\gamma$ And (A) | −1.5 | From dynamical parallax of $\gamma$ And (BC). B is main-sequence star of type A0. If absolute magnitude of +0.5 is assumed for B, then $M$ of $\gamma$ And (A) is −2.6 |
| | | K2+ | $\iota$ Aur | | |
| | | K2+ | $\gamma$ Aql | | |
| III | Normal giants | F9 | $\epsilon$ Hya | +0.1 | From dynamical parallax |
| | | G2 | $\alpha$ Aur | 0.0 | From spectroscopic and interferometric measures |
| | | G4+ | $\eta$ Peg | | |
| | | G6 | $\eta$ Her | | |
| | | K0 | $\beta$ Gem | +1.2 | Trigonometric parallax = 0".100 ± 10 |
| | | K1+ | $\alpha$ Boo | −0.1 | Trigonometric parallax = 0.085 ± 4 |
| | | K5 | $\alpha$ Tau | −0.6 | Trigonometric parallax = 0.046 ± 4 |
| | | K5+ | $\beta$ And | +0.6 | Trigonometric parallax = 0.044 ± 6 |
| IV | G5–K2 subgiants | G9 | $\eta$ Ser | +2.0 | Trigonometric parallax = 0.053 ± 7 |
| | | K1 | $\gamma$ Cep | +2.6 | Trigonometric parallax = 0.067 ± 5 |
| V | Main-sequence stars | F4 | $\iota$ Peg | +3.4 | Trigonometric parallax = 0.077 ± 6 |
| | | F6+ | $\pi^3$ Ori | +3.8 | Trigonometric parallax = 0.128 ± 5 |
| | | F8 | $\beta$ Vir | +3.8 | Trigonometric parallax = 0.101 ± 7 |
| | | F9+ | $\eta$ Cas (A) | +4.9 | Trigonometric parallax = 0.182 ± 5 |
| | | G1+ | $\delta$ Tri | +5.0 | Trigonometric parallax = 0.098 ± 6 |
| | | G8 | $\mu$ Cas | +5.8 | Trigonometric parallax = 0.130 ± 6 |
| | | G8+ | $\sigma$ Dra | +6.1 | Trigonometric parallax = 0.181 ± 4 |
| | | K2 | $\epsilon$ Eri | +6.2 | Trigonometric parallax = 0.305 ± 7 |
| | | K3 | HR 8832 | +6.5 | Trigonometric parallax = 0.146 ± 6 |
| | | K5+ | 61 Cyg A | +8.0 | Trigonometric parallax = 0.299 ± 3 |

* *Ap. J.*, **86**, 575, 1937.

† Half-tenths of spectral classes are designated by plus signs (+).

‡ Absolute magnitude may be affected by interstellar absorption.

§ *Lick Obs. Bull.*, **14**, 170, 1930.

TABLE 4

SPECTRAL TYPES AND COLOR INDICES OF STANDARD STARS

| No. | Star | α 1900 | δ 1900 | $m_H$ | $Sp_H$ | $Sp_Y$ | w | R | $L_Y$ | $C_B$ w |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | β Cas | 0h 4m | +58°.6 | 2.42 | F5 | F2 | (3) | 2 | III | −0.45 (1) |
| 2 | δ And | 0 34 | 30.3 | 3.49 | K2 | K3 | (3) | 0 | III | + .07 (2) |
| 3 | α Cas | 0 35 | 56.0 | 2.47 | K0 | K1 | (6) | 1 | II–III | .00 (1) |
| 4 | η Cas A | 0 43 | 57.3 | 3.64 | F8 | F9+ | (6) | 3 | V | − .37 (2) |
| 5 | μ Cas | 1 2 | 54.4 | 5.26 | G5 | G8 | (6) | 4 | V | − .31 (1) |
| 6 | β And | 1 4 | 35.1 | 2.37 | Ma | K5+ | (2) | 0 | III | + .26 (1) |
| 7 | α UMi | 1 23 | 88.8 | 2.12R | F8 | F7 | (4) | 2 | Ib | − .33 (8) |
| 8 | η Psc | 1 26 | 14.8 | 3.72 | G5 | G8+ | (2) | 1 | III | − .13 (1) |
| 9 | α Tri | 1 47 | 29.1 | 3.58 | F5 | F5+ | (3) | 1 | III | − .41 (1) |
| 10 | α Ari | 2 2 | 23.0 | 2.23 | K2 | K1+ | (5) | 2 | III | .00 (1) |
| 11 | δ Tri | 2 11 | 33.8 | 5.07 | G0 | G1+ | (3) | 1 | V | − .36 (3) |
| 12 | θ Per | 2 37 | 48.8 | 4.22 | F8 | F7+ | (3) | 1 | V | − .39 (1) |
| 13 | η Per | 2 43 | 55.5 | 3.93 | K0 | K2 | (4) | 1 | I | + .21 (1) |
| 14 | α Cet | 2 57 | 3.7 | 2.82 | Ma | M0 | (2) | 1 | III | + .29 (1) |
| 15 | ι Per | 3 2 | 49.2 | 4.17 | G0 | F9+ | (4) | 1 | V | − .35 (2) |
| 16 | α Per | 3 17 | 49.5 | 1.90 | F5 | F5 | (5) | 1 | Ib | − .40 (2) |
| 17 | ο Tau | 3 19 | 8.7 | 3.80 | G5 | G8 | (2) | 2 | III | − .20 (2) |
| 18 | ν Per | 3 38 | 42.3 | 3.93 | F5 | F4 | (3) | 0 | Ib | − .41 (2) |
| 19 | γ Tau | 4 14 | 15.4 | 3.86 | K0 | G9+ | (2) | 3 | III | − .11 (1) |
| 20 | δ Tau | 4 17 | 17.3 | 3.93 | K0 | G9 | (2) | 2 | III | − .11 (1) |
| 21 | ε Tau | 4 23 | 19.0 | 3.63 | K0 | K0 | (3) | 0 | III | − .11 (1) |
| 22 | α Tau | 4 30 | 16.3 | 1.06 | K5 | K5 | (6) | 1 | III | + .25 (2) |
| 23 | $\pi^3$ Ori | 4 44 | 6.8 | 3.31 | F8 | F6+ | (3) | 1 | V | − .44 (1) |
| 24 | ι Aur | 4 51 | 33.0 | 2.90 | K2 | K2+ | (3) | 1 | II | + .23 (1) |
| 25 | ε Aur | 4 55 | 43.7 | 3.3R | F5 | F4 | (3) | 1 | Ia | − .44 (2) |
| 26 | α Aur | 5 9 | 45.9 | 0.21R | G0 | G2 | (6) | 3 | III | − .25 (1) |
| 27 | λ Aur | 5 12 | 40.0 | 4.85 | G0 | G0 | (3) | 2 | V | − .32 (1) |
| 28 | $\chi^1$ Ori | 5 49 | 20.3 | 4.62 | F8 | G0 | (3) | 2 | V | − .36 (1) |
| 29 | α Ori | 5 50 | 7.4 | 0.9R | Ma | (M1) | (3) | 1 | I | + .34 (5) |
| 30 | δ Aur | 5 51 | 54.3 | 3.88 | K0 | G7 | (2) | 0 | III | − .11 (3) |
| 31 | ε Gem | 6 38 | 25.2 | 3.18 | G5 | G7+ | (4) | 3 | I | + .15 (8) |
| 32 | ξ Gem | 6 40 | 13.0 | 3.40 | F5 | F5+ | (3) | 1 | III | − .41 (1) |
| 33 | ι Gem | 7 20 | 28.0 | 3.89 | K0 | G9+ | (3) | 1 | III | − .12 (2) |
| 34 | α CMi | 7 34 | 5.5 | 0.48 | F5 | F4+ | (6) | 3 | V | − .43 (1) |
| 35 | κ Gem | 7 38 | 24.6 | 3.68 | G5 | G7 | (2) | 2 | III | − .13 (1) |
| 36 | β Gem | 7 39 | 28.3 | 1.21 | K0 | K0 | (6) | 3 | III | − .09 (3) |
| 37 | β Cnc | 8 11 | 9.5 | 3.76 | K2 | K4 | (3) | 0 | III | + .21 (1) |
| 38 | ο UMa | 8 22 | 61.1 | 3.47 | G0 | G7 | (6) | 2 | III | − .21 (1) |
| 39 | ε Hya | 8 42 | 6.8 | 3.48R | F8 | F9 | (2) | 1 | III | − .31 (1) |
| 40 | ζ Hya | 8 50 | 6.3 | 3.30 | K0 | G9 | (4) | 2 | III | − .07 (1) |
| 41 | HR 3579 | 8 54 | 42.2 | 4.09 | F5 | F5+ | (3) | 1 | V | − .43 (2) |
| 42 | α Lyn | 9 15 | 34.8 | 3.30 | K5 | K5+ | (4) | 1 | III | + .28 (2) |
| 43 | θ UMa | 9 26 | 52.1 | 3.26 | F8 | F6+ | (2) | 1 | III | − .40 (2) |
| 44 | 11 LMi | 9 30 | 36.3 | 5.48 | K0 | G8+ | (2) | 1 | V | − .21 (1) |
| 45 | ε Leo | 9h 40m | +24°.2 | 3.12 | G0 | G1 | (4) | 2 | II | −0.22 (3) |

TABLE 4—*Continued*

| No. | Star | α 1900 | δ 1900 | $m_H$ | $Sp_H$ | $Sp_Y$ | w | R | $L_Y$ | $C_B$ | w |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 46.... | 40 Leo | 10h14m | +20°.0 | 4.97 | F5 | F7+ | (2) | 1 | V | −0.37 | (1) |
| 47.... | γ Leo | 10 15 | +20.4 | 2.61R | Ko | K2 | (2) | 0 | III | .00 | (1) |
| 48.... | μ UMa | 10 16 | +42.0 | 3.21 | K5 | K5+ | (3) | 1 | III | + .25 | (4) |
| 49.... | 36 UMa | 10 24 | +56.5 | 4.84 | F5 | F6 | (2) | 0 | V | − .39 | (4) |
| 50.... | 46 LMi | 10 48 | +34.8 | 3.92 | Ko | K1 | (2) | 0 | III | − .08 | (1) |
| 51.... | α UMa | 10 58 | +62.3 | 1.95 | Ko | Ko+ | (3) | 1 | III | − .07 | (6) |
| 52.... | ψ UMa | 11 4 | +45.0 | 3.15 | Ko | K1 | (2) | 0 | III | − .03 | (1) |
| 53.... | ξ UMa | 11 13 | +32.1 | 4.41R | Go | F9+ | (4) | 1 | V | − .40 | (1) |
| 54.... | ν UMa | 11 13 | +33.6 | 3.71 | Ko | K3+ | (2) | 1 | III | + .16 | (1) |
| 55.... | χ UMa | 11 41 | +48.3 | 3.85 | Ko | Ko+ | (3) | 1 | III | + .03 | (1) |
| 56.... | β Vir | 11 46 | + 2.3 | 3.80 | F8 | F8 | (6) | 2 | V | − .37 | (2) |
| 57.... | β CVn | 12 29 | +41.9 | 4.32 | Go | Go | (3) | 2 | V | − .36 | (2) |
| 58.... | ε Vir | 12 57 | +11.5 | 2.95 | Ko | G8+ | (5) | 3 | III | − .14 | (1) |
| 59.... | β Com | 13 7 | +28.4 | 4.32 | Go | F9 | (4) | 2 | V | − .36 | (2) |
| 60.... | τ Boo | 13 43 | +18.0 | 4.51 | F5 | F6 | (3) | 0 | V | − .41 | (1) |
| 61.... | η Boo | 13 50 | +18.9 | 2.80 | Go | F9 | (5) | 5 | III | − .33 | (2) |
| 62.... | α Boo | 14 11 | +19.7 | 0.24 | Ko | K1+ | (6) | 1 | III | + .05 | (7) |
| 63.... | θ Boo | 14 22 | +52.3 | 4.06 | F8 | F7+ | (4) | 2 | V | − .41 | (2) |
| 64.... | ρ Boo | 14 28 | +30.8 | 3.78 | Ko | K2 | (2) | 0 | III | + .09 | (1) |
| 65.... | ξ Boo | 14 47 | +19.5 | 4.64R | G5 | Ko | (3) | 1 | V | − .25 | (3) |
| 66.... | β UMi | 14 51 | +74.6 | 2.24 | K5 | K4 | (3) | 0 | III | + .20 | (2) |
| 67.... | β Boo | 14 58 | +40.8 | 3.63 | G5 | G7 | (5) | 2 | III | − .12 | (2) |
| 68.... | δ Boo | 15 12 | +33.7 | 3.54 | Ko | G9+ | (2) | 1 | III | − .14 | (3) |
| 69.... | ι Dra | 15 23 | +59.3 | 3.47 | Ko | K1+ | (3) | 1 | III | + .02 | (4) |
| 70.... | α Ser | 15 39 | + 6.7 | 2.75 | Ko | K1+ | (2) | 1 | III | + .03 | (2) |
| 71.... | γ Ser | 15 52 | +16.0 | 3.86 | F5 | F6+ | (6) | 4 | V | − .43 | (2) |
| 72.... | δ Oph | 16 9 | − 3.4 | 3.03 | Ma | Mo | (3) | 1 | III | + .27 | (1) |
| 73.... | η Dra | 16 23 | +61.7 | 2.89 | G5 | G9+ | (2) | 1 | III | − .17 | (4) |
| 74.... | β Her | 16 26 | +21.7 | 2.81 | Ko | G8+ | (2) | 3 | III | − .15 | (2) |
| 75.... | ζ Her | 16 38 | +31.8 | 3.00 | Go | G1 | (6) | 4 | V | − .32 | (1) |
| 76.... | η Her | 16 40 | +39.1 | 3.61 | Ko | G6 | (4) | 6 | III | − .17 | (1) |
| 77.... | κ Oph | 16 53 | + 9.5 | 3.42 | Ko | K1+ | (1) | ... | III | − .01 | (1) |
| 78.... | π Her | 17 12 | +36.9 | 3.36 | K5 | K3 | (3) | 2 | II–III | + .16 | (1) |
| 79.... | β Dra | 17 28 | +52.4 | 2.99 | Go | F9 | (4) | 1 | I | − .18 | (2) |
| 80.... | β Oph | 17 39 | + 4.6 | 2.94 | Ko | K2 | (4) | 1 | III | + .03 | (1) |
| 81.... | μ Her | 17 43 | +27.8 | 3.48 | G5 | G5+ | (3) | 2 | V | − .26 | (2) |
| 82.... | ξ Dra | 17 52 | +56.9 | 3.90 | Ko | K1 | (2) | 0 | III | − .04( | 1) |
| 83.... | θ Her | 17 53 | +37.3 | 3.99 | Ko | Ko+ | (2) | 1 | II | + .12 | (1) |
| 84.... | ξ Her | 17 54 | +29.3 | 3.82 | Ko | G8 | (4) | 2 | III–IV | − .15 | (1) |
| 85.... | γ Dra | 17 54 | +51.5 | 2.42 | K5 | K5 | (5) | 1 | III | + .22 | (2) |
| 86.... | 70 Oph | 18 0 | + 2.5 | 4.07R | Ko | Ko+ | (2) | 1 | V | − .20 | (1) |
| 87.... | η Ser | 18 16 | − 2.9 | 3.42 | Ko | G9 | (3) | 3 | IV | − .15 | (1) |
| 88.... | χ Dra | 18 23 | +72.7 | 3.69 | F8 | F6 | (5) | 1 | V | − .40 | (2) |
| 89.... | 110 Her | 18 41 | +20.5 | 4.26 | F5 | F6 | (1) | ... | V | − .43 | (1) |
| 90.... | δ Dra | 19h13m | +67°.5 | 3.24 | Ko | Ko | (2) | 2 | III | −0.13 | (1) |

TABLE 4—*Continued*

| No. | Star | α 1900 | δ 1900 | $m_H$ | $Sp_H$ | $Sp_Y$ | w | R | $L_Y$ | $C_B$ | w |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 91.... | κ Cyg | $19^h15^m$ | +53°.2 | 3.98 | K0 | G8+ | (3) | 2 | III–IV | −0.15 | (1) |
| 92.... | σ Dra | 19 33 | +69.5 | 4.78 | K0 | G8+ | (4) | 3 | V | − .23 | (2) |
| 93.... | θ Cyg | 19 34 | +50.0 | 4.64 | F5 | F4 | (2) | 2 | V | − .44 | (1) |
| 94.... | γ Aql | 19 42 | +10.4 | 2.80 | K2 | K2+ | (4) | 1 | II | + .25 | (4) |
| 95.... | ε Dra | 19 49 | +70.0 | 3.99 | K0 | G7 | (2) | 2 | III | − .21 | (1) |
| 96.... | β Aql | 19 50 | + 6.2 | 3.90 | K0 | K0 | (5) | 2 | IV–V | − .21 | (1) |
| 97.... | γ Sge | 19 54 | +19.2 | 3.71 | K5 | K5+ | (4) | 1 | III | + .25 | (1) |
| 98.... | γ Cyg | 20 19 | +39.9 | 2.32 | F8 | F8 | (5) | 2 | I*b* | − .30 | (6) |
| 99.... | β Del | 20 33 | +14.3 | 3.72 | F5 | F4 | (2) | 2 | III | − .40 | (1) |
| 100.... | ε Cyg | 20 42 | +33.6 | 2.64 | K0 | K0 | (4) | 0 | III–IV | − .09 | (2) |
| 101.... | η Cep | 20 43 | +61.5 | 3.59 | K0 | G9+ | (5) | 3 | IV–V | − .17 | (1) |
| 102.... | ξ Cyg | 21 1 | +43.5 | 3.92 | K5 | K3+ | (4) | 1 | I | + .30 | (1) |
| 103.... | 61 Cyg | 21 2 | +38.3 | 5.57[R] | K5 | K5+[R] | (2) | ... | V | + .04 | (2) |
| 104.... | ζ Cyg | 21 9 | +29.8 | 3.40 | K0 | G9 | (2) | 2 | II | − .10 | (1) |
| 105.... | ε Peg | 21 39 | + 9.4 | 2.54 | K0 | K1+ | (6) | 1 | I | + .25 | (1) |
| 106.... | α Aqr | 22 1 | − 0.8 | 3.19 | G0 | F9 | (4) | 2 | I | − .16 | (1) |
| 107.... | ι Peg | 22 2 | +24.9 | 3.96 | F5 | F4 | (4) | 1 | V | − .45 | (1) |
| 108.... | ζ Cep | 22 7 | +57.7 | 3.62 | K0 | K1 | (6) | 1 | I | + .24 | (1) |
| 109.... | η Peg | 22 38 | +29.7 | 3.10 | G0 | G4+ | (4) | 4 | III | − .21 | (2) |
| 110.... | μ Peg | 22 45 | +24.1 | 3.67 | K0 | G7+ | (2) | 1 | III | − .18 | (1) |
| 111.... | ι Cep | 22 46 | +65.7 | 3.68 | K0 | K0 | (2) | 0 | III | − .10 | (2) |
| 112.... | HR 8832 | 23 9 | +56.6 | 5.65 | K2 | K3 | (2) | 0 | V | − .08 | (1) |
| 113.... | γ Psc | 23 12 | + 2.7 | 3.85 | K0 | G7 | (5) | 3 | III | − .20 | (1) |
| 114.... | γ Cep | $23^h35^m$ | +77°.1 | 3.42 | K0 | K1 | (3) | 0 | IV | −0.10 | (1) |

NOTES TO TABLE 4

7. α UMi The light and color index are both variable to a slight degree.

25. ε Aur. Visual magnitude at maximum.

26. α Aur. The apparent magnitude of each of the two components is considered to be 1.0.

29. α Ori. Semiregular variable.

39. ε Hya. Apparent magnitude of primary; secondary is probably too faint to affect the color or spectral type.

47. γ Leo. Apparent magnitude of primary; secondary may affect spectral type and color to a slight degree.

53. ξ UMa. The two brightest components of this multiple system contribute to both the observed spectral type and the color index. The apparent magnitude is that of the brighter star.

65. ξ Boo. The fainter component probably has little effect on the observed spectral type and color index.

86. 70 Oph. The fainter component may affect the observed spectral type and color index slightly.

103. 61 Cyg. The apparent magnitude of the brighter component is given. As the photoelectric color index applies to both stars considered as one, the catalogue type is a mean value weighted according to the apparent magnitudes of the components.

Stars having composite spectra have been omitted.

what greater than those computed from the internal agreement. The values of the mean error have correspondingly been rounded off arbitrarily in the following manner:

| | F4–G0 | G1–G9 | K0–K5 |
|---|---|---|---|
| Adjusted mean error of one plate | ±1.0 | ±1.5 | ±0.5 |

It is considerably more difficult to give a measure of the accuracy of the luminosity classification because of the fact that the mean error varies with spectral type and also with luminosity. Because of the many marked spectral characteristics of the supergiants, the probability is exceedingly small that a good spectrogram of a supergiant will be classified in a group of lower luminosity. At the other extreme is the case of the separation of the subgiants and dwarfs of types G8–K1; the mean error of one plate here is of the order of one-half of a luminosity class. A rough set of estimates of the mean error of the luminosity classification between successive groups for one good spectrogram, in units of a luminosity class, is shown in the accompanying table.

| Class | F4–F7 | F8–G1 | G2–G6 | G7–K1 | K2–K5 |
|---|---|---|---|---|---|
| I–II | ... | $\frac{1}{4}$ | ... | $\frac{1}{4}$ | $\frac{1}{4}$ |
| II–III | ... | $\frac{1}{2}$ | ... | $\frac{1}{4}$ | $\frac{1}{4}$ |
| I–III | $\frac{1}{4}$ | ... | ... | ... | ... |
| III–IV | ... | ... | $\frac{1}{4}$ | $\frac{1}{4}$ | ... |
| III–V | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | ... | $<\frac{1}{4}$ |
| IV–V | ... | ... | ... | $\frac{1}{2}$ | ... |

The mean error of a single observation of photoelectric color index for stars in Table 4 is ±0.016 mag.; this value was determined by Schlesinger's method from stars which were observed between three and six times. The average mean error for a tabular color index when the number of observations is taken into account is ±0.013 mag.

4. The photoelectric color indices of the three most important luminosity groups are plotted against Yerkes spectral types in Figure 1. In addition to the stars in Table 4 the colors of the brightest B and A stars are plotted against their Draper spectral types; all

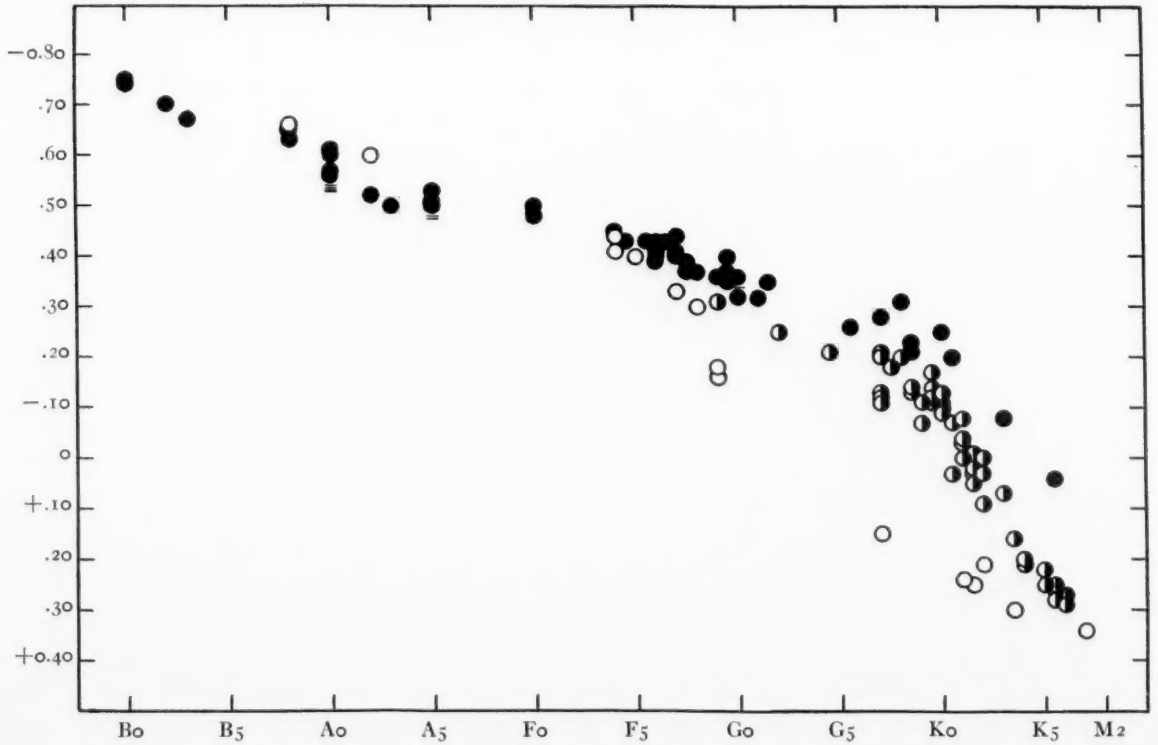


FIG. 1.—Bottlinger's photoelectric color indices and Yerkes spectral types for stars in Table 4. Open circles are supergiants, half-closed circles ordinary giants, and closed circles main-sequence stars. The diagram also includes color indices and Draper spectral types of B stars brighter than magnitude 2.0 and A stars brighter than 3.0. The radius of the circles is equal to the average mean error of the color indices. Several normal giants of types F2–F6 have not been plotted, for reasons of clarity.

B stars brighter than apparent magnitude 2.0 and all A stars brighter than 3.0 for which Bottlinger measured color indices have been included. The supergiants (which are denoted by open circles) show little or no systematic difference in color from main-sequence stars (closed circles) for types earlier than F5; after this they diverge rapidly in color and reach a maximum difference of more than four-tenths of a magnitude on Bottlinger's scale at G8–K0. The divergence in color of the supergiants from the normal giants (half-closed circles) also passes through a maximum in the nighborhood of types G8–K0 and decreases rapidly after that point; by the time K5 is reached the difference is relatively small.

A general inspection of Figure 1 shows the smooth progression of change in color index with spectral type for each luminosity group; the dispersion in color within a luminosity class at any given spectral type is low, and it is noteworthy that there is not a single case, for stars of types later than F8, of a dwarf which is as red as a giant of the same type or of a giant which is as red as a supergiant. Of particular interest is the low dispersion in color for dwarfs of types F4–G1 and for normal giants of types K4–M0. It should not be assumed, however, that the actual dispersion in color is necessarily higher for stars having the same absolute magnitude and spectral type in the region G6–K1; the scatter of the plotted points is considerably greater, but this is due, at least to a considerable extent, to the large variation of color with absolute magnitude for stars of these spectral types, to the somewhat lower accuracy of the classification at G5–G9, and to the more rapid change of color with spectral type. This tends to superpose an appreciable additional accidental error on the average errors of classification of spectral type and of the color indices. For the F stars the total range in color with luminosity is much less; in the case of the K4–M0 stars the problem is simplified by the absence of subgiants and by the small difference in color between normal giants and supergiants.

For these reasons a test of the accuracy with which the intrinsic color index is defined by spectral type and luminosity is most readily made for the F4–G1 stars and for the giants of types K4–M0. Because of the considerably greater number of F4–G1 main-sequence stars, they form the most suitable group for investigation. The color

indices and spectral types of these objects are plotted on a larger scale in Figure 2. The radius of the plotted circles corresponds to the average mean error of the color indices.

The color indices are well represented by a straight line whose equation is

$$C_D = -0^m429 + 0^m015(S - F5) \qquad G2 \gtrless S \gtrless F4 ,$$

where $C_D$ is the color index of a main-sequence star on Bottlinger's scale, $S$ is the Yerkes spectral type, and $(S - F5)$ is expressed in

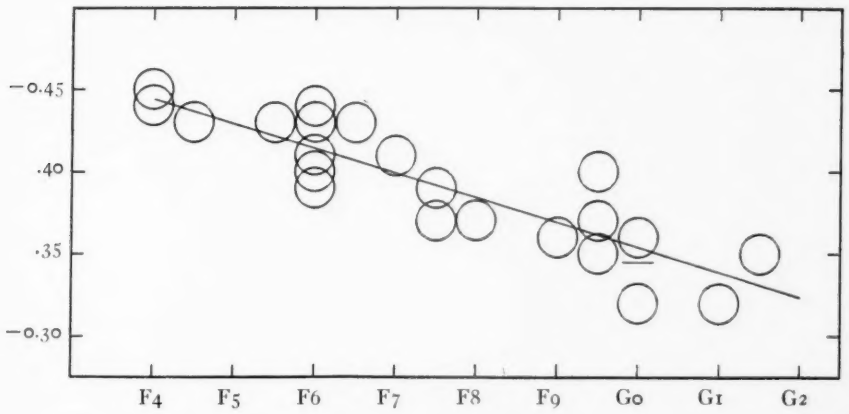


FIG. 2.—Photoelectric color indices and Yerkes spectral types of main-sequence stars of types F4–G1. The straight line has the equation:

$$C_D = -0^M429 + 0^M015\ (S-F5).$$

units of tenths of a spectral class. The mean error of a plotted point from this straight line is ±0.018 mag. Of the twenty-three points plotted, two have the maximum residual of 0.04 mag. The representation of the observed colors of the main-sequence stars by their spectral types is, therefore, very close.

The average mean error of a value of Yerkes spectral type for the stars in Figure 2 is ±0.6$S$. As the change of color index is 0.014 mag. per tenth of a spectral class, the mean error of classification is equivalent to a mean error of ±0.008 mag. in color index. The mean error of a catalogue value of the color index is ±0.013 mag. This corresponds to a combined mean error of ±0.015 mag., as compared with

an observed mean error from Figure 2 of ±0.019 mag. The difference in the two values is slight; when it is considered that corrections for components of spectroscopic binaries and close visual binaries have not been made and that two systems of internal mean errors are being combined, the agreement can be considered to be satisfactory. The conclusion seems justified that within the errors of observation there is no real dispersion in the intrinsic colors of the F4–G1 main sequence stars. The color indices of these objects seem to be uniquely defined by their spectral types.

TABLE 5

COLORS OF G9–K0 NORMAL GIANTS

| Star | Yerkes Spectral Type | Color Index | Δ |
|---|---|---|---|
| ζ Hya | G9 (4) | −0.07 (1) | 0.05 |
| δ Tau | G9 (2) | .11 (1) | .01 |
| γ Tau | G9+ (2) | .11 (1) | .01 |
| ι Gem | G9+ (3) | .12 (2) | .00 |
| η Dra | G9+ (2) | .17 (4) | .05 |
| δ Boo | G9+ (2) | .14 (3) | .02 |
| ε Tau | K0 (3) | .11 (1) | .01 |
| ι Cep | K0 (2) | .10 (2) | .02 |
| δ Dra | K0 (2) | .13 (1) | .01 |
| β Gem | K0 (7) | −0.09 (3) | 0.03 |

The dispersion is correspondingly low in the case of the K4–M0 normal giants. The subgiants, which introduce additional uncertainty in the neighborhood of K0, are absent, and the supergiants are only slightly redder than the normal giants.

We shall finally consider the normal giants of spectral types G9–K0 to obtain an approximate value of the dispersion in color at an unfavorable position, that is, where the change of color with luminosity is rapid. The stars are listed in Table 5, together with the deviation of their color indices from the arithmetical mean. The mean error of an individual color index in Table 5 is ±0.03 mag.—about 50 per cent higher than in the case of the F4–G1 main-sequence stars. The increase in the mean error can be accounted for by two factors: (1) the change of color index with spectral type is more rapid for K giants than for the F5–G0 dwarfs; (2) the rapid change of color index with luminosity introduces an additional de-

gree of uncertainty. The first factor is the more important, as the value of $dC/dS$ for normal giants at K0 is about 0.04 mag.

5. The evidence as a whole, therefore, seems to point to the conclusion reached for the F4–G1 dwarfs: if the accuracy of the observations is taken into account, the color indices of the stars of spectral types F4–M0 are uniquely determined by their spectral type and luminosity. This conclusion makes way for several interesting practical applications of calibration diagrams similar to Figures 1 and 2. If, for near-by stars, or stars in regions known to be free from absorbing clouds, spectral types and luminosities are available of the accuracy of those in the present paper, then the color indices of these stars may be predicted with considerable accuracy. In the case of the G5–K1 stars these color indices should be of the same order of accuracy as those obtained from a comparison of photographic and photovisual catalogues of high weight; for the F4–G1 dwarfs and for the K4–M0 giants the accuracy of the predicted colors should be somewhat higher. The conclusion is especially important in the case of the F5–G0 dwarfs; because of their low absolute magnitude, difficulties with selective absorption should not be encountered until stars fainter than the eighth apparent magnitude are considered. They are thus capable of serving as "calibrated lamps" whose spectral energy distribution may be predicted with high accuracy and which can be used with considerable confidence as standards of color.

Some of the spectrograms were obtained by Dr. P. C. Keenan, Mr. E. Ebbighausen, and Mr. G. W. Wares. I am indebted to Miss Edith Kellman for computational assistance and for making the drawings.

YERKES OBSERVATORY
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# IONIZATION AND RECOMBINATION IN THE THEORY OF STELLAR ABSORPTION LINES AND NEBULAR LUMINOSITY

S. CHANDRASEKHAR

ABSTRACT

In this paper the effect of collisions and photo-ionizations on monochromatic radiative equilibrium is considered. A formalism is developed which includes the theory of stellar absorption lines (Strömgren) and the theory of nebular luminosity (Zanstra) as two limiting cases. A second approximation for the nebular problem is also sketched. The mathematical analysis is devoted to the treatment of the three-state problem including collisions. When collisions can be neglected, the approximate equations of transfer used admit of two flux integrals and two $K$-integrals.

1. *Introduction.*—The theory of the formation of stellar absorption lines hinges on the problem of writing down the appropriate equation of radiative transfer. If we can write down the equation of transfer, then we have, in principle, solved the problem. But in the considerations leading to the equation of transfer, we can follow one of two methods: either argue in terms of macroscopic notions, e.g., the mass absorption and emission coefficients, etc., or in terms of microscopic notions, e.g., the Einstein transition probabilities, effective cross-section for captures, etc. The two methods, if properly handled, should of course lead to identical results; but a comparison of the two methods is often useful. Thus, in the so-called "combined Schuster-Schwarzschild problem" it is assumed that we can characterize the material by a mass absorption coefficient, $\kappa$, and a mass scattering coefficient, $l$. The equation of transfer is then written down in the form

$$\frac{dI_\nu}{\rho ds} = -(\kappa_\nu + l_\nu)I_\nu + \kappa_\nu B_\nu + l_\nu J_\nu , \tag{1}$$

where $I_\nu$ is the specific intensity at frequency $\nu$, $B_\nu$ the Planck function, and

$$J_\nu = \int I_\nu \frac{d\omega}{4\pi} = \frac{1}{2}\int I_\nu \sin\theta d\theta . \tag{2}$$

In writing down (1), it is assumed that the emission consists of two parts: (*a*) the Kirchhoff emission, $\kappa_\nu B_\nu$, and (*b*) the emission due to scattering, of amount $l_\nu J_\nu$. On the other hand, as E. A. Milne[1] has shown, we can derive an equation similar to (1) by an entirely different type of discussion, which does not involve an appeal to Kirchhoff's law. The assumptions, (*a*) a Maxwellian distribution of velocities and (*b*) the effect of collisions in exciting and de-exciting the atoms, lead to the equation of transfer

$$\frac{dI_{\nu_{12}}}{dt_\nu} = -I_{\nu_{12}} + \frac{J_\nu + \eta_{\nu_{12}} B_{\nu_{12}}}{1 + \eta_{\nu_{12}}}, \tag{3}$$

where $dt_{\nu_{12}}$ is the optical depth measured in the direction of $s$,

$$\eta_{\nu_{12}} = \frac{b_{12}}{B_{12}} \frac{1}{B_{\nu_{12}}}, \tag{4}$$

where $b_{12}$ and $B_{12}$ are the probabilities of an atom in state 1 being excited to state 2 by collision or by the absorption of radiation of appropriate frequency. If we write

$$\eta_{\nu 12} = \frac{\kappa_\nu}{l_\nu} \tag{5}$$

and

$$dt_\nu = (\kappa_\nu + l_\nu)\rho ds, \tag{5'}$$

we see that (1) and (3) are formally equivalent. We shall return to the discussion of the physical meaning of (5) and (5'); but it is already clear that Milne's method, if properly interpreted, will remove some of the obscurity involved in the appeal to Kirchhoff's law in the usual treatment.

However, the theory of the formation of stellar absorption lines in the frame either of (1) or of (3) encounters difficulties—among others, the matter of the central intensities.

The physical foundations have been analyzed by B. Strömgren,[2]

[1] *M.N.*, **88**, 493, 1928.

[2] *Zs. f. Ap.*, **10**, 237, 1935; *Handbuch der Ap.* **7**, 221–235, 1936.

who incorporates into the theory at an early stage the influence of electron captures and photo-ionizations. Strömgren's theory is entirely satisfactory, but at the same time it would be of some interest to examine whether Milne's method can be suitably generalized to lead to results equivalent to those obtained by Strömgren's method. We shall see that this is possible.

There is another aspect to this problem. As is well known, Zanstra's theory of nebular luminosity utilizes the processes of electron captures and ionization as a means for the conversion of the ultraviolet energy beyond the head of the Lyman series in the incident starlight into energy mainly in Lyman $\alpha$. The problem of radiative transfer associated with the Zanstra theory has been studied by V. A. Ambarzumian and the writer.[3] The question now arises whether we cannot construct a scheme which will include, as limiting cases, Strömgren's equation of transfer at one end and the equations of radiative transfer for the Zanstra theory at the other end. The main object of this paper is to supply such a scheme.

2. *Definitions and fundamental equations.*—The method consists in extending Milne's procedure to include cyclical transitions between three states, one of which we eventually identify with the continuum.

First let us consider the case of three discrete levels of statistical weights, $g_1$, $g_2$, $g_3$. We introduce the Einstein coefficients $A_{nm}$, $B_{nm}$, $B_{mn}$, defined with respect to the intensity of radiation. We have the usual relations

$$\frac{A_{nm}}{B_{mn}} = \frac{2h\nu_{mn}^3}{c^2}\frac{g_m}{g_n}, \tag{6}$$

$$\frac{B_{nm}}{B_{mn}} = \frac{g_m}{g_n}. \tag{7}$$

We shall use the abbreviation

$$\sigma_{mn} = \frac{2h\nu_{mn}^3}{c^2}. \tag{8}$$

[3] S. Chandrasekhar, *Zs. f. Ap.*, **9**, 266, 1935.

If $a_{mn}(\nu)$ is the atomic absorption coefficient for the atom in the state $m$ for radiation of frequency $\nu$, then

$$\int a_{mn}(\nu)d\nu = \frac{B_{mn}h\nu_{mn}}{4\pi}, \tag{9}$$

the integral on the left-hand side being taken through the narrow range of frequencies about $\nu_{mn}$ within which the atom can absorb. We write (9) in the form

$$a_{mn}\Delta\nu_{mn} = \frac{B_{mn}h\nu_{mn}}{4\pi}, \tag{10}$$

where $a_{mn}$ is the mean atomic absorption coefficient in the interval $\Delta\nu_{mn}$. We shall next introduce the probability coefficients $b_{mn}$, $a_{nm}$, to take into account the effect of collisions in exciting and de-exciting the atoms in the states $m$ and $n$, $(m < n)$—more precisely, the probability that in unit time an atom in state $m$, $(n)$, is excited (de-excited) by collisions of the first (second) kind is $b_{mn}$, $(a_{nm})$. If we assume that a Maxwellian distribution of the atomic velocities corresponding to a certain temperature $T$ is maintained, then we should have

$$\frac{b_{mn}}{a_{nm}} = \frac{g_n}{g_m} e^{-h\nu_{mn}/kT}. \tag{11}$$

Now, in a steady state the number of atoms, $n_1$, $n_2$, $n_3$, in each of the states must remain constant. Thus, the constancy of the number of atoms in the ground state 1 yields

$$n_1(B_{12}J_{12} + b_{12} + B_{13}J_{13} + b_{13}) - n_2(A_{21} + B_{21}J_{12} + a_{21}) - n_3(A_{31} + B_{31}J_{13} + a_{31}) = 0, \tag{12}$$

where the $J$'s are used to denote

$$J_{mn} = \int I_{mn}\frac{d\omega}{4\pi} = \frac{1}{2}\int I_{mn}\sin\theta d\theta. \tag{13}$$

Using (6), (7), (8), and (11), we can re-write (12) in the form

$$\left.\begin{aligned} n_1(B_{12}J_{12} + b_{12} + B_{13}J_{13} + b_{13}) - n_2\frac{g_1}{g_2}B_{12}\left(\sigma_{12} + J_{12} + \frac{b_{12}}{B_{12}}e^{h\nu_{13}/kT}\right) \\ - n_3\frac{g_1}{g_3}B_{13}\left(\sigma_{13} + J_{13} + \frac{b_{13}}{B_{13}}e^{h\nu_{13}/kT}\right) = 0\,. \end{aligned}\right\} \tag{14}$$

Similarly, the constancy of the number of atoms in state 3 gives

$$\left.\begin{aligned} n_1(B_{13}J_{13} + b_{13}) + n_2(B_{23}J_{23} + b_{23}) - n_3\left\{\frac{g_1}{g_3}B_{13}\left(\sigma_{13} + J_{13} + \frac{b_{13}}{B_{13}}e^{h\nu_{13}/kT}\right)\right. \\ \left. + \frac{g_2}{g_3}B_{23}\left(\sigma_{23} + J_{23} + \frac{b_{23}}{B_{23}}e^{h\nu_{23}/kT}\right)\right\} = 0\,. \end{aligned}\right\} \tag{15}$$

As may easily be verified, (14) and (15) automatically provide for the constancy of the number of atoms in the second state. We introduce the abbreviations

$$\hat{\omega}_{mn} = \frac{b_{mn}}{B_{mn}}\,,\ (m < n)\,, \tag{16}$$

$$\beta_{mn} = \hat{\omega}_{mn}\,e^{h\nu_{mn}/kT},\ (m < n)\,, \tag{17}$$

$$\xi_{mn} = \sigma_{mn} + \beta_{mn} + J_{mn}\,. \tag{18}$$

Equations (14) and (15) now take the forms

$$n_1\{B_{12}(J_{12} + \hat{\omega}_{12}) + B_{13}(J_{13} + \hat{\omega}_{13})\} - n_2\frac{g_1}{g_2}B_{12}\xi_{12} - n_3\frac{g_1}{g_3}B_{13}\xi_{13} = 0\,, \tag{19}$$

$$n_1B_{13}(J_{13} + \hat{\omega}_{13}) + n_2B_{23}(J_{23} + \hat{\omega}_{23}) - n_3\left\{\frac{g_1}{g_3}B_{13}\xi_{13} + \frac{g_2}{g_3}B_{23}\xi_{23}\right\} = 0\,, \tag{20}$$

which are our equations of radiative equilibrium.

So far, we have restricted ourselves to three discrete states. We now wish to make state 3 the "continuum," i.e., the transitions $1 \to 3$, $2 \to 3$, now correspond to photo-ionization. As Ambarzu-

mian[4] has shown, we can treat the continuum on the same footing as a discrete state, provided we define for the weight $g_3$ the expression

$$g_3 = \frac{g_+}{N_e}\frac{(2\pi m_e kT)^{3/2}}{h^3}, \tag{21}$$

where $g_+$ is the statistical weight of the normal state of the ionized atom and $N_e$ is the number of free electrons per cubic centimeter. Furthermore, $B_{13}$ and $B_{23}$ are now no longer atomic constants but are the probability coefficients of the photoelectric transitions defined in an appropriate way.

From (19) and (20), we derive

$$\left.\begin{aligned} &\frac{n_1}{\frac{g_1}{g_2}B_{12}\xi_{12}\left\{\frac{g_1}{g_3}B_{13}\xi_{13}+\frac{g_2}{g_3}B_{23}\xi_{23}\right\}+\frac{g_1}{g_3}B_{13}\xi_{13}B_{23}(J_{23}+\hat{\omega}_{23})} \\ &= \frac{n_2}{\{B_{12}(J_{12}+\hat{\omega}_{12})+B_{13}(J_{13}+\hat{\omega}_{13})\}\left\{\frac{g_1}{g_3}B_{13}\xi_{13}+\frac{g_2}{g_3}B_{23}\xi_{23}\right\}-\frac{g_1}{g_3}B_{13}\xi_{13}B_{13}(J_{13}+\hat{\omega}_{13})} \\ &= \frac{n_3}{B_{23}(J_{23}+\hat{\omega}_{23})\{B_{12}(J_{12}+\hat{\omega}_{12})+B_{13}(J_{13}+\hat{\omega}_{13})\}+\frac{g_1}{g_2}B_{12}\xi_{12}B_{13}(J_{13}+\hat{\omega}_{13})} \end{aligned}\right\} \tag{22}$$

3. *The equations of transfer.*—The equations of transfer are easily seen to be (cf. Milne, *loc. cit.*)

$$\frac{dI_{12}}{\alpha_{12}ds} = -\left(n_1 - n_2\frac{g_1}{g_2}\right)I_{12} + \frac{g_1}{g_2}n_2\sigma_{12}, \tag{23}$$

$$\frac{dI_{13}}{\alpha_{13}ds} = -\left(n_1 - n_3\frac{g_1}{g_3}\right)I_{13} + \frac{g_1}{g_3}n_3\sigma_{13}, \tag{24}$$

$$\frac{dI_{23}}{\alpha_{23}ds} = -\left(n_2 - n_3\frac{g_2}{g_3}\right)I_{23} + \frac{g_2}{g_3}n_3\sigma_{23}, \tag{25}$$

[4] *M.N.*, **95**, 469, 1935.

where the $\alpha_{mn}$'s are given by (10). The optical depths $dt_{12}$, $dt_{13}$, and $dt_{23}$ can be defined by

$$dt_{12} = \alpha_{12}\left(n_1 - n_2\frac{g_1}{g_2}\right)ds = \frac{B_{12}h\nu_{12}\left(n_1 - n_2\dfrac{g_1}{g_2}\right)}{4\pi\Delta\nu_{12}}\,ds\,, \tag{26}$$

$$dt_{13} = \alpha_{13}\left(n_1 - n_3\frac{g_1}{g_3}\right)ds = \frac{B_{13}h\nu_{13}\left(n_1 - n_3\dfrac{g_1}{g_3}\right)}{4\pi\Delta\nu_{13}}\,ds\,, \tag{27}$$

$$dt_{23} = \alpha_{23}\left(n_2 - n_3\frac{g_2}{g_3}\right)ds = \frac{B_{23}h\nu_{23}\left(n_2 - n_3\dfrac{g_2}{g_3}\right)}{4\pi\Delta\nu_{23}}\,ds\,. \tag{28}$$

It should be noticed that our optical depths, $dt_{mn}$, are measured in the direction in which the variation of $I$ is considered. The equations (23), (24), and (25) now take the forms:

$$\frac{dI_{12}}{dt_{12}} = -I_{12} + \frac{n_2\dfrac{g_1}{g_2}}{n_1 - n_2\dfrac{g_1}{g_2}}\,\sigma_{12}\,, \tag{29}$$

$$\frac{dI_{13}}{dt_{13}} = -I_{13} + \frac{n_3\dfrac{g_1}{g_3}}{n_1 - n_3\dfrac{g_1}{g_3}}\,\sigma_{13}\,, \tag{30}$$

$$\frac{dI_{23}}{dt_{23}} = -I_{23} + \frac{n_2\dfrac{g_2}{g_3}}{n_2 - n_3\dfrac{g_2}{g_3}}\,\sigma_{23}\,. \tag{31}$$

It will be noticed that (29), (30), and (31) involve the $n$'s only through the ratios $(n_1/n_2)$, $(n_1/n_3)$, $(n_2/n_3)$, respectively. The solution (22) of the equations of radiative equilibrium provides the values of the required ratios in terms of the $J$'s; and equations (29), (30), and (31) form a system of simultaneous differential equations. To evaluate the terms involving the $n$'s we shall adopt a method of approximation introduced by Ambarzumian. We regard $J_{12}/\sigma_{12}$ and $J_{23}/\sigma_{23}$ as quantities of the first order of smallness, while we

regard $J_{13}/\sigma_{13}$ and $J_{12}J_{23}/\sigma_{12}\sigma_{23}$ as quantities of the second order of smallness. The justification for this is found in the circumstance that, near the star, $J_{12}/\sigma_{12}$ and $J_{23}/\sigma_{13}$ are of the order of magnitude $e^{-h\nu_{12}/kT}$ and $e^{-h\nu_{23}/kT}$, respectively, while $J_{13}/\sigma_{13}$, for instance, is of the order of magnitude $e^{-h(\nu_{12}+\nu_{23})/kT}$. On the other hand, if we are far away from the star, the effect of the dilution factor will be to make our approximation sufficient for practical purposes. Let

$$\hat{\omega}_{mn} = \eta_{mn} B_{\nu_{mn}} , \tag{32}$$

where $B_{\nu_{mn}}$ is the Planck function:

$$B_{\nu_{mn}} = \sigma_{mn}(e^{h\nu_{mn}/kT} - 1)^{-1} . \tag{33}$$

By (18) and (32) we have

$$\xi_{mn} = \sigma_{mn} + J_{mn} + \eta_{mn}\sigma_{mn}(1 - e^{-h\nu_{mn}/kT})^{-1} . \tag{34}$$

Hence, we have

$$\xi_{12} = \sigma_{12}(1 + \eta_{12}) + \text{first-order quantities} , \tag{35}$$

$$\xi_{23} = \sigma_{23}(1 + \eta_{23}) + \text{first-order quantities} , \tag{36}$$

$$\xi_{13} = \sigma_{13}(1 + \eta_{13}) + \text{second-order quantities} . \tag{37}$$

We evaluate the expressions occurring on the right-hand sides of the equations of transfer, using (22) consistent with our method of approximation. After some rather lengthy calculations, it is found that

$$\frac{n_2 \frac{g_1}{g_2}}{n_1 - n_2 \frac{g_1}{g_2}} = \mathbf{J}_{12}\left[1 - \frac{\mu_{13}\mu_{23}}{\mu_{12}(\mu_{13} + \mu_{23})}\mathbf{J}_{23}\right] + \frac{\mu_{13}\mu_{23}}{\mu_{12}(\mu_{13} + \mu_{23})}\mathbf{J}_{13} , \tag{38}$$

$$\frac{n_3 \frac{g_1}{g_3}}{n_1 - n_3 \frac{g_1}{g_3}} = \mathbf{J}_{13}\left[1 - \frac{\mu_{23}}{\mu_{13} + \mu_{23}}\right] + \frac{\mu_{23}}{\mu_{13} + \mu_{23}}\mathbf{J}_{23}\mathbf{J}_{12} , \tag{39}$$

$$\frac{n_3 \frac{g_2}{g_3}}{n_2 - n_3 \frac{g_2}{g_3}} = \mathbf{J}_{23}\left[1 - \frac{\mu_{13}}{\mu_{13} + \mu_{23}}\right] + \frac{\mu_{13}}{\mu_{13} + \mu_{23}}\frac{\mathbf{J}_{13}}{\mathbf{J}_{12}} , \tag{40}$$

where

$$\mu_{12} = g_1 B_{12}\sigma_{12}(1 + \eta_{12}) , \tag{41}$$

$$\mu_{13} = g_1 B_{13}\sigma_{13}(1 + \eta_{13}) , \tag{42}$$

$$\mu_{23} = g_2 B_{23}\sigma_{23}(1 + \eta_{23}) . \tag{43}$$

$$\mathbf{J}_{12} = \frac{J_{12} + \eta_{12} B_{\nu_{12}}}{\sigma_{12}(1 + \eta_{12})} , \tag{44}$$

$$\mathbf{J}_{13} = \frac{J_{13} + \eta_{13} B_{\nu_{13}}}{\sigma_{13}(1 + \eta_{13})} , \tag{45}$$

$$\mathbf{J}_{23} = \frac{J_{23} + \eta_{23} B_{\nu_{23}}}{\sigma_{23}(1 + \eta_{23})} . \tag{46}$$

Let

$$p = \frac{\mu_{13}}{\mu_{13} + \mu_{23}} ; \qquad 1 - p = \frac{\mu_{23}}{\mu_{13} + \mu_{23}} . \tag{47}$$

Then

$$\frac{\mu_{13}\mu_{23}}{\mu_{12}(\mu_{13} + \mu_{23})} = (1 - p)\frac{\mu_{13}}{\mu_{12}} . \tag{47'}$$

By (46) and (47) we can re-write (38), (39), and (40) in the forms

$$\frac{n_2 \frac{g_1}{g_2}}{n_1 - n_2 \frac{g_1}{g_2}} = \mathbf{J}_{12}\left[1 - (1 - p)\frac{\mu_{13}}{\mu_{12}}\mathbf{J}_{23}\right] + (1 - p)\frac{\mu_{13}}{\mu_{12}}\mathbf{J}_{13} , \tag{48}$$

$$\frac{n_3 \frac{g_1}{g_3}}{n_1 - n_3 \frac{g_1}{g_3}} = p\mathbf{J}_{13} + (1 - p)\mathbf{J}_{23}\mathbf{J}_{12} , \tag{49}$$

$$\frac{n_3 \frac{g_2}{g_3}}{n_2 - n_3 \frac{g_2}{g_3}} = (1 - p)\mathbf{J}_{23} + p\frac{\mathbf{J}_{13}}{\mathbf{J}_{12}} . \tag{50}$$

The equations of transfer now reduce to

$$\frac{dI_{12}}{dt_{12}} = -I_{12} + \left[1 - (1-p)\frac{\mu_{13}}{\mu_{12}}\mathbf{J}_{23}\right]\sigma_{12}\mathbf{J}_{12} + (1-p)\sigma_{12}\frac{\mu_{13}}{\mu_{12}}\mathbf{J}_{13}, \quad (51)$$

$$\frac{dI_{13}}{dt_{13}} = -I_{13} + p\sigma_{13}\mathbf{J}_{13} + (1-p)\sigma_{13}\mathbf{J}_{23}\mathbf{J}_{12}, \quad (52)$$

$$\frac{dI_{23}}{dt_{23}} = -I_{23} + (1-p)\sigma_{23}\mathbf{J}_{23} + p\sigma_{23}\frac{\mathbf{J}_{13}}{\mathbf{J}_{12}}. \quad (53)$$

We shall now proceed to consider two limiting cases of the foregoing system of equations.

4. *The formation of absorption lines: comparison with Strömgren's results.*—According to equations (44) and (51),

$$\frac{dI_{12}}{dt_{12}} = -I_{12} + (1-\epsilon_{12})\frac{J_{12} + \eta_{12}B_{\nu_{12}}}{1+\eta_{12}} + \sigma_{12}\epsilon_{12}\frac{\mathbf{J}_{13}}{\mathbf{J}_{23}}, \quad (54)$$

where we have used $\epsilon_{12}$ to denote

$$\epsilon_{12} = (1-p)\frac{\mu_{13}}{\mu_{12}}\mathbf{J}_{23} = \frac{\mu_{23}\mu_{13}}{\mu_{12}(\mu_{13}+\mu_{23})}\mathbf{J}_{23}. \quad (55)$$

We re-write (54) in the form

$$(1+\eta_{12})\frac{dI_{12}}{dt_{12}} = -(1+\eta_{12})I_{12} + (1-\epsilon_{12})(J_{12} + \eta_{12}B_{\nu_{12}}) + \epsilon_{12}\sigma_{12}(1+\eta_{12})\frac{\mathbf{J}_{13}}{\mathbf{J}_{23}}, \quad (56)$$

or, again, after some further rearrangement of the terms,

$$(1+\eta_{12})\frac{dI_{12}}{dt_{12}} = -(1+\eta_{12})I_{12} + (1-\epsilon_{12})J_{12} + \eta_{12}B_{\nu_{12}} + \epsilon_{12}Q_{12}B_{\nu_{12}}, \quad (57)$$

where we have used $Q_{12}$ to denote

$$Q_{12} = \frac{\sigma_{12}(1+\eta_{12})\mathbf{J}_{13} - \eta_{12}B_{\nu_{12}}\mathbf{J}_{23}}{B_{\nu_{12}}\mathbf{J}_{23}}. \quad (58)$$

The equation of transfer (57), which we have derived, includes the effect of collisions and of the cyclical transitions (resulting from ionization and captures) on monochromatic radiative equilibrium. Consequently, it can form the basis of an analysis of the formation of absorption lines, and as such we can regard $\eta$ as small, compared with unity. This simplifies our expression for $Q_{12}$. We can write, according to (45), (46), and (58),

$$Q_{12} \simeq \frac{\sigma_{12}}{B_{\nu_{12}}} \frac{J_{13}}{\sigma_{13}} \frac{\sigma_{23}}{J_{23}} . \tag{59}$$

Further, we have

$$\frac{J_{13}}{\sigma_{13}} \simeq e^{-h\nu_{13}/kT_c} ; \qquad \frac{J_{23}}{\sigma_{23}} \simeq e^{-h\nu_{23}/kT_c} ; \qquad \frac{B_{\nu_{12}}}{\sigma_{12}} \simeq e^{-h\nu_{12}/kT} , \tag{60}$$

where we interpret $T_c$ as the color temperature of the radiation. On the other hand, the Planck function $B_{\nu_{12}}$ is defined with respect to the temperature of the material. Hence, we have

$$Q_{12} \simeq e^{h\nu_{12}\left(\frac{1}{kT}-\frac{1}{kT_c}\right)} . \tag{61}$$

We can also simplify our expression for $\epsilon_{12}$. According to equations (41), (42), (43), (47), and (55), we have

$$\epsilon_{12} \simeq \frac{g_2 B_{23}\sigma_{23} \cdot g_1 B_{13}\sigma_{13}}{g_1 B_{12}\sigma_{12}(g_1 B_{13}\sigma_{13} + g_2 B_{23}\sigma_{23})} \frac{J_{23}}{\sigma_{23}} . \tag{62}$$

Now, we can further assume that $B_{23} \ll B_{13}$, since the total probability of photo-ionizations from the excited state can be neglected, compared with the probability of photo-ionization from the ground state. We can thus write

$$\epsilon_{12} \simeq \frac{g_2}{g_1 B_{12}\sigma_{12}} B_{23} J_{23} , \tag{63}$$

or, using (6), we have

$$\epsilon_{12} \simeq \frac{B_{23} J_{23}}{A_{21}} . \tag{64}$$

The term $B_{23}J_{23}$ has a simple physical meaning: it represents the probability of photo-ionization of atoms in state 2 owing to the incident radiation field. If we denote this probability by $C_{2f}$, we have

$$\epsilon_{12} \simeq \frac{C_{2f}}{A_{21}} . \tag{65}$$

Now, Strömgren writes the equation of transfer in the form (*loc. cit.*, Eq. [37])

$$\frac{dI}{\rho ds} = -(\kappa_\nu + l_\nu)I_\nu + (1 - \epsilon)l_\nu J_\nu + \kappa_\nu B_\nu + \epsilon Q l_\nu B_\nu . \tag{66}$$

We verify that Strömgren's $\epsilon$ and $Q$ are identical with our $\epsilon_{12}$ and $Q_{12}$ (cf. our equations [65] and [61] with Strömgren's [18] and the equation at the top of p. 253 of his paper). We thus see that our equation of transfer (57) will become identical with Strömgren's (66) if we write in (57)

$$dt_{12} = (\kappa_{\nu_{12}} + l_{\nu_{12}})\rho ds , \tag{67}$$

$$\eta_{12} = \frac{\kappa_{\nu_{12}}}{l_{\nu_{12}}} . \tag{68}$$

We thus see that the correspondence between our treatment and Strömgren's problem is of the same nature as the correspondence between Milne's treatment and the combined Schuster-Schwarzschild problem.

In comparing our method with Strömgren's, the following points should be noted.

1. In our method we consider (strictly speaking) only a one-constituent atmosphere, the relevant physical processes being collisions, photo-ionizations, recombinations, and radiative resonance transitions. Consequently, the only processes which give rise to general opacity in our model are collisions of the first and second kinds. It is in this sense that equation (67) has to be interpreted.

2. Our method does not take into account the contributions to the general opacity arising from factors other than collisions and, further, entirely ignores the presence of other constituents in the at-

mosphere. In Strömgren's method, however, it is assumed that *if* the general opacity, $\kappa_\nu$, owing to all factors and all the constituents of the atmosphere, is known, *then* the appropriate emission is $\kappa_\nu B_\nu(T)$, where $T$ is the temperature of the material.

3. The origin of the identity of our $\epsilon_{12}$ and $Q_{12}$ with Strömgren's $\epsilon$ and $Q$ is to be found in the circumstance that in both the treatments the effect of photo-ionizations on monochromatic equilibrium is considered in physically equivalent terms.

It is thus clear that the fact that in the model case we have considered—namely, the three-state problem including collisions—our method leads to an equation of transfer identical with that derived by Strömgren, can be interpreted as implying the validity of the form in which use is made of Kirchhoff's law in Strömgren's method.

5. *The equations of the planetary nebulae.*—Returning to equations (51) and (52), we see that in the case of high dilution of radiation and in the absence of collisions we can write them in the forms

$$\frac{dI_{12}}{dt_{12}} = -I_{12} + J_{12} + \sigma_{12}(1-p)\frac{g_1 B_{13}\sigma_{13}}{g_1 B_{12}\sigma_{12}}\frac{J_{13}}{\sigma_{13}}, \tag{69}$$

$$\frac{dI_{13}}{dt_{13}} = -I_{13} + pJ_{13}, \tag{70}$$

where

$$p = \frac{g_1 B_{13}\sigma_{13}}{g_1 B_{13}\sigma_{13} + g_2 B_{23}\sigma_{23}}. \tag{71}$$

Equation (69) can be further simplified by using (10):

$$\frac{dI_{12}}{dt_{12}} = -I_{12} + J_{12} + (1-p)\frac{a_{13}\Delta\nu_{13}}{a_{12}\Delta\nu_{12}}\frac{\nu_{12}}{\nu_{13}}J_{13}. \tag{72}$$

These are precisely the equations of transfer in the form used by Ambarzumian and the writer. The probability $p$ introduced in that theory is identical with our present $p$, defined according to (71). We thus see that our formalism developed in sections 2 and 3 of this paper includes the theory of stellar absorption lines and the theory of nebular luminosity as two limiting cases.

6. *The equations of the planetary nebulae: a second approximation.* —In the absence of collisions, our equations of transfer are (cf. Eqs. [51], [52], and [53]):

$$\frac{dI_{12}}{dt_{12}} = -I_{12} + J_{12}\left[1 - (1-p)\frac{\mu_{13}}{\mu_{12}}\frac{J_{23}}{\sigma_{23}}\right] + (1-p)\frac{\mu_{13}}{\mu_{12}}\frac{\sigma_{12}}{\sigma_{13}}J_{13}\,, \quad (73)$$

$$\frac{dI_{13}}{dt_{13}} = -I_{13} + pJ_{13} + (1-p)\frac{\sigma_{13}}{\sigma_{12}\sigma_{23}}J_{12}J_{23}\,, \quad (74)$$

$$\frac{dI_{23}}{dt_{23}} = -I_{23} + (1-p)J_{23} + p\frac{\sigma_{13}\sigma_{23}}{\sigma_{13}}\frac{J_{13}}{J_{12}}\,, \quad (75)$$

where $p$ is defined as in equation (71) and

$$\mu_{13} = g_1 B_{13}\sigma_{13}\,; \qquad \mu_{12} = g_1 B_{12}\sigma_{12}\,. \quad (76)$$

Hence,

$$\frac{\mu_{13}}{\mu_{12}} = \frac{B_{13}\sigma_{13}}{B_{12}\sigma_{12}}\,. \quad (77)$$

Let

$$\gamma = \frac{\sigma_{13}}{\sigma_{12}\sigma_{23}}\,. \quad (78)$$

Then

$$\frac{dI_{12}}{dt_{12}} = -I_{12} + J_{12}\left[1 - \gamma(1-p)\frac{B_{13}}{B_{12}}J_{23}\right] + (1-p)\frac{B_{13}}{B_{12}}J_{13}\,, \quad (79)$$

$$\frac{dI_{13}}{dt_{13}} = -I_{13} + pJ_{13} + \gamma(1-p)J_{12}J_{23}\,, \quad (80)$$

$$\frac{dI_{23}}{dt_{23}} = -I_{23} + (1-p)J_{23} + \frac{p}{\gamma}\frac{J_{13}}{J_{12}}\,. \quad (81)$$

It should be remembered that in the foregoing equations the optical depths are measured in the direction in which the variation of $I$ is considered.

From our definitions of $dt_{12}$, $dt_{13}$, $dt_{23}$ we verify that

$$\frac{dt_{13}}{dt_{23}} = \frac{B_{13}\nu_{13}\Delta\nu_{23}}{B_{23}\nu_{23}\Delta\nu_{13}} \frac{n_1 - n_3 \dfrac{g_1}{g_3}}{n_2 - n_3 \dfrac{g_2}{g_3}}, \tag{82}$$

or, using (49) and (50) and remembering that for the case under consideration $\eta = 0$, we have

$$\frac{dt_{13}}{dt_{23}} = \frac{B_{13}\nu_{13}\Delta\nu_{23}}{B_{23}\nu_{23}\Delta\nu_{13}} \frac{g_1}{g_2} \frac{\sigma_{12}}{J_{12}} = \frac{g_1}{g_2} \frac{a_{13}}{a_{23}} \frac{\sigma_{12}}{J_{12}}. \tag{83}$$

Hence, equation (81) can be re-written as

$$\sigma_{12} \frac{dI_{23}}{dt_{13}} = \frac{g_2 a_{23}}{g_1 a_{13}} \{ - I_{23}J_{12} + (1 - p)J_{23}J_{12} + p\gamma^{-1}J_{13} \}. \tag{84}$$

We now see that equations (79), (80), and (84) form a consistent system of equations. The right-hand side of equation (79) contains terms of the first and the second order of smallness, while the right sides of equations (80) and (84) contain terms only of the second order of smallness. Consequently, it would be sufficient to find the dominant term in $dt_{13}/dt_{12}$. We verify that, to the order of accuracy we are working with, we have

$$\frac{dt_{13}}{dt_{12}} = \frac{B_{13}\nu_{13}\Delta\nu_{12}}{B_{12}\nu_{12}\Delta\nu_{13}} = \frac{a_{13}}{a_{12}}. \tag{85}$$

We introduce the abbreviations

$$\frac{\nu_{12}\Delta\nu_{13}}{\nu_{13}\Delta\nu_{12}} = \delta_1 ; \qquad \frac{\nu_{23}\Delta\nu_{13}}{\nu_{13}\Delta\nu_{23}} = \delta_2 ; \tag{86}$$

$$\frac{a_{13}}{a_{12}} = \frac{1}{\delta_1} \frac{B_{13}}{B_{12}} = q_1 ; \qquad \frac{g_1}{g_2} \frac{a_{13}}{a_{23}} = q_2 . \tag{87}$$

Considering the case of material stratified in parallel planes and denoting $d\tau$ to measure the optical depth in (1, 3) normal to the plane of stratification, the equations of transfer reduce to

$$q_1 \cos\theta \frac{dI_{12}}{d\tau} = -I_{12} + J_{12}[1 - \gamma\delta_1 q_1(1-p)J_{23}] + \delta_1 q_1(1-p)J_{13}, \quad (88)$$

$$\cos\theta \frac{dI_{13}}{d\tau} = -I_{13} + pJ_{13} + \gamma(1-p)J_{12}J_{23}, \quad (89)$$

$$q_2\sigma_{12}\cos\theta \frac{dI_{23}}{d\tau} = -I_{23}J_{12} + (1-p)J_{23}J_{12} + p\gamma^{-1}J_{13}. \quad (90)$$

These equations admit of three first integrals. Multiply the equations successively by $\frac{1}{2}\sin\theta d\theta$ and $\frac{1}{2}\sin\theta\cos\theta d\theta$, and integrate from 0 to $\pi$. Let

$$F_{mn} = 2\int_0^\pi I_{mn}\sin\theta\cos\theta d\theta, \quad (91)$$

$$K_{mn} = \frac{1}{2}\int_0^\pi I_{mn}\sin^2\theta\cos\theta d\theta. \quad (92)$$

Then

$$\frac{1}{4}\frac{dF_{12}}{d\tau} = \delta_1(1-p)[J_{13} - \gamma J_{23}J_{12}], \quad (93)$$

$$\frac{1}{4}\frac{dF_{13}}{d\tau} = (1-p)[\gamma J_{23}J_{12} - J_{13}], \quad (94)$$

$$q_2\sigma_{12}\frac{1}{4}\frac{dF_{23}}{d\tau} = p\gamma^{-1}[J_{13} - \gamma J_{23}J_{12}]. \quad (95)$$

Also

$$q_1\frac{dK_{12}}{d\tau} = -\tfrac{1}{4}F_{12}, \quad (96)$$

$$\frac{dK_{13}}{d\tau} = -\tfrac{1}{4}F_{13}, \quad (97)$$

$$q_2\sigma_{12}\frac{dK_{23}}{d\tau} = -\tfrac{1}{4}F_{23}J_{12}. \quad (98)$$

From (93), (94), and (95) we derive

$$\frac{dF_{12}}{d\tau} = -\delta_1 \frac{dF_{13}}{d\tau}, \tag{99}$$

$$\frac{dF_{23}}{d\tau} = -\frac{p}{\gamma q_2 \sigma_{12}(1-p)} \frac{dF_{13}}{d\tau}. \tag{100}$$

We first notice that, according to (47), (76), (78), (83), and (86), we have

$$\frac{p}{\gamma q_2 \sigma_{12}(1-p)} = \delta_2 . \tag{101}$$

Hence, (99) and (100) yield the two flux integrals

$$F_{12} + \delta_1 F_{13} = \text{Constant} = S_{12} + \delta_1 S_{13}, \tag{102}$$

$$F_{23} + \delta_2 F_{13} = \text{Constant} = S_{23} + \delta_2 S_{23}, \tag{103}$$

where we have used $S_{mn}$ to denote the "incident" flux. From (102) and (103) we derive

$$\delta_2(F_{12} - S_{12}) = \delta_1(F_{23} - S_{23}) . \tag{104}$$

We now derive a $K$-integral: from (96) and (97), we have

$$q_1 \frac{dK_{12}}{d\tau} + \delta_1 \frac{dK_{13}}{d\tau} = -\tfrac{1}{4}(F_{12} + \delta_1 F_{13}), \tag{105}$$

or, by (102),

$$\frac{d}{d\tau}\{q_1 K_{12} + \delta_1 K_{13}\} = -\tfrac{1}{4}(S_{12} + \delta_1 S_{13}) . \tag{106}$$

Hence,

$$q_1 K_{12} + \delta_1 K_{13} = -\tfrac{1}{4}(S_{12} + \delta_1 S_{13})\tau + \text{constant} . \tag{107}$$

We easily verify that (107) is equivalent to an expression which the author had derived in his paper on the radiative equilibrium of a planetary nebula (*loc. cit.*, Eq. [87]). However, we have now derived it under more general conditions.

It is possible to derive another approximate $K$-integral if

$$\delta_2 S_{12} = \delta_1 S_{23} . \tag{108}$$

Then, according to (104), $\delta_2 F_{12} = \delta_1 F_{23}$, and we can re-write (98) as

$$\delta_1 q_2 \sigma_{12} \frac{dK_{23}}{d\tau} = -\tfrac{1}{4} \delta_2 F_{12} J_{12} , \tag{109}$$

or, by (96),

$$\delta_1 q_2 \sigma_{12} \frac{dK_{23}}{d\tau} = \delta_2 q_1 \frac{dK_{12}}{d\tau} J_{12} . \tag{110}$$

If we make the approximation

$$J_{12} = 3K_{12} , \tag{111}$$

we have

$$\delta_1 q_2 \sigma_{12} \frac{dK_{23}}{d\tau} = \tfrac{3}{2} \delta_2 q_1 \frac{d}{d\tau} (K_{12}^2) , \tag{112}$$

or

$$\sigma_{12} K_{23} = \frac{3}{2} \frac{\delta_2 q_1}{\delta_1 q_2} [K_{12}^2 + \text{constant}] , \tag{113}$$

which can be written alternatively as

$$\frac{K_{23}}{\sigma_{23}} = \frac{3}{2} \frac{\delta_2 q_1}{\delta_1 q_2} \frac{\sigma_{12}}{\sigma_{23}} \left[ \left( \frac{K_{12}}{\sigma_{12}} \right)^2 + \text{constant} \right] . \tag{114}$$

Since $(K_{12}/\sigma_{12})^2$ is a quantity of the second order of smallness, while $(K_{23}/\sigma_{23})$ is one of the first order of smallness, it is apparent from (114) that, if we neglect quantities of the second order, then $K_{23}$ is a constant.

The integrals (102), (103), (107), and (113) have been derived from the equations of transfer (88), (89), and (90), which are approximations to the exact equations of the three-state problem.

However, Henyey has since shown that integrals corresponding to those we have derived can be isolated for the exact problem.

We shall next consider a little more closely the case $S_{12} = S_{23} = 0$, so that the integral (113) is valid. Let $S_{13}$ be the incident flux in the (1, 3) radiation. We can then write

$$F_{13} = S_{13}e^{-\tau} + \mathfrak{F}_{13}\,, \tag{115}$$

$$K_{13} = \tfrac{1}{4}S_{13}e^{-\tau} + \mathfrak{K}_{13}\,, \qquad J_{13} = \tfrac{1}{4}S_{13}e^{-\tau} + \mathfrak{J}_{13} \tag{116}$$

where $\mathfrak{F}_{13}$ and $\mathfrak{K}_{13}$ are defined for the scattered radiation. Our flux integrals can be written as

$$\delta_2 F_{12} = \delta_1\delta_2[S_{13}(1 - e^{-\tau}) - \mathfrak{F}_{13}] = \delta_1 F_{23}\,. \tag{117}$$

Furthermore, we have

$$q_1 K_{12} + \delta_1 K_{13} = -\tfrac{1}{4}\delta_1 S_{13}(\tau + C_1)\,, \tag{118}$$

where $C_1$ is a constant. From (93) and (96) we derive

$$q_1 \frac{d^2K_{12}}{d\tau^2} = \delta_1(1 - p)[\gamma J_{23}J_{12} - J_{13}]\,. \tag{119}$$

We shall use the approximation

$$J_{12} = 3K_{12}\,; \qquad J_{23} = 3K_{23}\,; \qquad \mathfrak{J}_{13} = 3\mathfrak{K}_{13}\,. \tag{120}$$

Then

$$q_1 \frac{d^2K_{12}}{d\tau^2} = 3\delta_1(1 - p)[3\gamma K_{23}K_{12} - K_{13} + \tfrac{1}{6}S_{13}e^{-\tau}]\,. \tag{121}$$

Using the $K$-integral in the form (118), the foregoing equation reduces to

$$\frac{d^2K_{12}}{d\tau^2} = 3(1 - p)\left[1 + \frac{3\delta_1\gamma}{q_1}K_{23}\right]K_{12} + \frac{3(1 - p)\delta_1 S_{13}}{4q_1} \times (\tau + \tfrac{2}{3}e^{-\tau} + C_1)\,. \tag{122}$$

Using our second $K$-integral and denoting by $C_2$ the constant of integration occurring in (113), we finally have

$$\frac{d^2K_{12}}{d\tau^2} = 3(1-p)\left[1 + \frac{9}{2}\frac{\delta_2}{q_2}\frac{\sigma_{13}}{\sigma_{23}}\left\{\left(\frac{K_{12}}{\sigma_{12}}\right)^2 + C_2\right\}\right]K_{12}$$

$$+ \frac{3(1-p)\delta_1 S_{13}}{4q_1}\left(\tau + \tfrac{2}{3}e^{-\tau} + C_1\right), \quad (123)$$

which is the fundamental differential equation of the problem. After solving (123) for $K_{12}$, we determine $F_{12}$ by using (96). Once $K_{12}$ and $F_{12}$ are known, all the other quantities can be derived successively, using the different integrals. Concrete applications of (123) will be found in a forthcoming paper.

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# REVIEWS

*Introduction to Physiological Optics.* By J. P. C. SOUTHALL. New York: Oxford University Press, 1937. Pp. 426. $5.50.

This very readable introduction to physiological optics is intended as much for laymen who wish to know something about vision as for students who need a broad treatment of a subject finding application in physics, medicine, optometry, ophthalmology, cinema technique, and illumination. Scientific workers who obtain data with visual instruments may find the book advantageous for recalling some of the vagaries of the eye. What little mathematics is used involves scarcely anything more difficult than the algebra of first-order optics and of colorimetry. All conclusions are stated clearly in words instead of symbolic equations. Considerable interest is provided by the frequent description of little experiments designed to show such points as the adaptation and accommodation of the eye, blur circles on the retina, the Purkinje effect, the blind spot, and various aspects of binocular vision, of rod versus cone vision, of contrast effects, and of color perception. A number of the classical experiments of Scheiner, Wollaston, Helmholtz, and others are included among these. The text is often varied by the introduction of appropriate passages from earlier books and papers by many authors. There is a profusion of drawings.

Chapter i describes the structure of the eye and some of the subjective phenomena of vision, including Fechner's law, which is here stated to be "more honored in the breach than in the observance." Chapter ii is a discussion of the imagery of the Gullstrand schematic eye. The next three chapters deal with the problem of correcting eye-glasses for the mobile eye afflicted with ametropia, presbyopia, or astigmatism. This section includes a discussion of the design of the *Punktal* lens, and has several pages on contact eyeglasses. Chapter vi treats in detail of the movements of the eyes in binocular vision, of depth perception, and of stereoscopy. Rod and cone vision are compared in the next chapter, which deals with dark adaptation, color sensitivity, and zonal sensitivity of the retina. Chapters viii and ix are devoted to color vision and color mixing. The Young-Helmholtz theory of color vision is described and criticized, and the theories of Hering and Ostwald are presented. The last chapter discusses persistence of vision, flicker phenomena, and after-images in monotone and in color.

Although the topics discussed are essentially the standard ones of physiological optics, the author's treatment combines an exceedingly interesting approach with judgment in selecting unhackneyed experiments and illustrations.

E. L. MCCARTHY